Permutation representations of finite groups and association schemes (Part 5)

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Review

For a commutative association scheme $\mathcal{R}$, the adjacency algebra $\mathbb{C}[\mathcal{R}]$ has two canonical bases $\{A_i\}$ and $\{E_i\}$, and they are related by the first and second eigenmatrices P, Q:

$$(A_0, \ldots, A_d) = (E_0, \ldots, E_d)P,$$

$$(E_0, \ldots, E_d) = \frac{1}{|\Omega|}(A_0, \ldots, A_d)Q,$$

$$\frac{Q_{ij}}{\operatorname{rank} E_j} = \frac{\overline{P_{ji}}}{k_i}.$$

The structure constants for the algebra $\mathbb{C}[\mathcal{R}]$ are

$$A_i A_j = \sum_k p_{ij}^k A_k \quad \text{(intersection numbers)},$$

$$E_i E_j = \sum_k q_{ij}^k E_k \quad \text{(Krein parameters)}.$$

Regular representations

The association scheme defined by the right regular representation $(G_R; G)$ is not commutative unless G is abelian, since its centralizer algebra is the algebra of left regular representation $\cong \mathbb{C}[G]$. For $g, \alpha, \beta \in G$,

$$
\begin{aligned}
(P_g)_{\alpha\beta} &= \delta_{\alpha g, \beta}, \\
(L_g)_{\alpha\beta} &= \delta_{g^{-1}\alpha, \beta} = \delta_{\alpha, g\beta}.
\end{aligned}
$$

Then for $g, h \in G$,

$$
\begin{aligned}
P_g P_h &= P_{gh}, \quad L_g L_h = L_{gh}, \\
L_g P_h &= P_h L_g.
\end{aligned}
$$

This defines an action and a permutation representation

$$
\begin{aligned}
G \times G \to \mathrm{Sym}(G) &\text{ by } (g, h) \mapsto (\alpha \mapsto g^{-1} \alpha h), \\
G \times G \to M_G(\mathbb{C}) &\text{ by } (g, h) \mapsto L_g P_h.
\end{aligned}
$$

2-orbits are edges in a normal Cayley graph

What are the 2-orbits of the action

$$G \times G \to \mathrm{Sym}(G) \text{ by } (g, h) \mapsto (\alpha \mapsto g^{-1}\alpha h).$$

Let $\mathcal{R} = G^2/(G \times G)$ be the set of 2-orbits. Then

$$\mathcal{R} \ni \exists R \ni (1, \beta), (1, \gamma) \iff \exists g \in G,\ g^{-1}\beta g = \gamma.$$

Let $C_0 = \{1\}$, $C_1, \ldots, C_d$ be the conjugacy classes of G, that is, the orbits of the conjugacy action $g \mapsto (\alpha \mapsto g^{-1}\alpha g)$. Then

$$\begin{aligned}\mathcal{R} &= \{\{(\alpha, \beta) \in G^2 \mid \alpha^{-1}\beta \in C_i\} \mid i = 0, \ldots, d\} \\ &= \{\{(\alpha, \alpha g) \mid \alpha \in G,\ g \in C_i\} \mid i = 0, \ldots, d\} \\ &= \{\bigcup_{g \in C_i} \{(\alpha, \alpha g) \mid \alpha \in G\} \mid i = 0, \ldots, d\}.\end{aligned}$$

$$A(R_i) = \sum_{g \in C_i} P_g \quad (i = 0, \ldots, d).$$

The center of $\mathbb{C}[G]$ (I)

$$\mathbb{C}[\mathcal{R}] = \mathrm{Span}\{\sum_{g \in C_i} P_g \mid i = 0, \ldots, d\},$$

and this is the center $Z(\mathbb{C}[G])$ of $\mathbb{C}[G] = \mathbb{C}[\{P_g \mid g \in G\}]$.

Recall from the double centralizer theorem,
$\mathbb{C}[\{P_g \mid g \in G\}]$ is the centralizer of $\mathbb{C}[\{L_g \mid g \in G\}]$,

$$\begin{aligned} A \in Z(\mathbb{C}[G]) &\iff A \text{ belongs to } \mathbb{C}[\{P_g \mid g \in G\}] \text{ and} \\ &\qquad A \text{ centralizes } P_g \ (g \in G) \\ &\iff A \text{ centralizes } L_g \text{ and } P_g \ (g \in G) \\ &\iff A \text{ commutes with the action of } G \times G \text{ on } G \\ &\iff A \in \mathbb{C}[\mathcal{R}]. \end{aligned}$$

The center of $\mathbb{C}[G]$ (II)

From part 3,

$$\mathbb{C}[G] \cong \begin{bmatrix} \ddots & & \\ & \boxed{M_{m_i}(\mathbb{C}) \otimes I_{m_i}} & \\ & & \ddots \end{bmatrix}.$$

Thus

$$Z(\mathbb{C}[G]) \cong \begin{bmatrix} \ddots & & \\ & \boxed{\mathbb{C}I_{m_i} \otimes I_{m_i}} & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \boxed{\mathbb{C}I_{m_i^2}} & \\ & & \ddots \end{bmatrix}.$$

$$\text{primitive idempotents are } E_i = U \begin{bmatrix} 0 & & \\ & \boxed{I_{m_i^2}} & \\ & & 0 \end{bmatrix} U^*.$$

$Z(\mathbb{C}[G])$ as the adjacency algebra

We have a commutative association scheme with adjacency matrices

$$A_i = A(R_i) = \sum_{g \in C_i} P_g \quad (i = 0, \ldots, d),$$

where $C_0 = \{1\}$, $C_1, \ldots, C_d$ are the conjugacy classes of G. Its adjacency algebra $\mathbb{C}[\mathcal{R}] = \mathrm{Span}\{A_i \mid i = 0, \ldots, d\}$ has

$$\text{primitive idempotents } E_i = U \begin{bmatrix} 0 & & \\ & \boxed{I_{m_i^2}} & \\ & & 0 \end{bmatrix} U^*,$$

so

$$A_j = \sum_i U \begin{bmatrix} 0 & & \\ & \boxed{P_{ij} I_{m_i^2}} & \\ & & 0 \end{bmatrix} U^*.$$

Characters of a finite group

From part 3,

$$\mathbb{C}[G] \cong \begin{bmatrix} \ddots & & \\ & \boxed{M_{m_i}(\mathbb{C}) \otimes I_{m_i}} & \\ & & \ddots \end{bmatrix}_{i=0,1,\dots,d}.$$

Note $d+1 = \dim Z(\mathbb{C}[G]) =$ the number of conjugacy classes.

For each i, the homomorphism $\Delta_i : \mathbb{C}[G] \to M_{m_i}(\mathbb{C})$ is an irreducible representation of $\mathbb{C}[G]$. Restricting Δ_i to G, we obtain an irreducible representation

$$\phi_i : G \to GL_{m_i}(\mathbb{C}).$$

Composing ϕ_i with tr, we obtain a function

$$\chi_i : G \to \mathbb{C}, \quad g \mapsto \operatorname{tr} \phi_i(g) = \operatorname{tr} \Delta_i(P_g),$$

called an irreducible characters of G. Note $\chi_i(1) = m_i$.

A character is a class function

$$\chi_i : G \to \mathbb{C}, \quad g \mapsto \operatorname{tr} \phi_i(g).$$

The function χ_i takes a constant value on each conjugacy class C_j.

$$\begin{aligned}
g_1, g_2 \in C_j \implies & \exists h \in G,\ g_1 = h^{-1} g_2 h \\
\implies & \chi_i(g_1) = \operatorname{tr} \phi_i(h^{-1} g_2 h) \\
& \quad = \operatorname{tr}(\phi_i(h)^{-1} \phi_i(g_2) \phi_i(h)) \\
& \quad = \operatorname{tr} \phi_i(g_2) \\
& \quad = \chi_i(g_2).
\end{aligned}$$

In other words,

$$\chi_i(g) = \chi_i(h^{-1} g h) \quad (g, h \in G).$$

What is P_{ij}? Eigenvalues of a normal Cayley graph

$$A_j = \sum_i U \begin{bmatrix} 0 & & \\ & \boxed{P_{ij} I_{m_i^2}} & \\ & & 0 \end{bmatrix} U^* \rightsquigarrow \Delta_i(A_j) = P_{ij} I_{m_i}.$$

Thus

$$\begin{aligned} n_i P_{ij} &= \operatorname{tr} \Delta_i(A_j) = \operatorname{tr} \Delta_i(\sum_{g \in C_j} P_g) \\ &= \sum_{g \in C_j} \operatorname{tr} \phi_i(g) = \sum_{g \in C_j} \chi_i(g) \\ &= |C_j| \chi_i(g_j), \end{aligned}$$

where $g_j \in C_j$. Since $n_i = \chi_i(1)$, we have

$$P_{ij} = \frac{|C_j| \chi_i(g_j)}{\chi_i(1)} \quad (g_j \in C_j).$$

This is an algebraic integer, as an eigenvalue of A_j.

The trivial character

Recall that every adjacency algebra $\mathbb{C}[\mathcal{R}]$ of an association scheme on Ω has a rank 1 primitive idempotent $E_0 = \frac{1}{|\Omega|}J$.

In our case now,

$$E_0 = \frac{1}{|G|}J.$$

$$\begin{aligned} P_{0j}E_0 = A_jE_0 &= \frac{1}{|G|}\sum_{g\in C_j} P_gJ = \frac{1}{|G|}\sum_{g\in C_j} J = \frac{|C_j|}{|G|}J \\ &= |C_j|E_0. \end{aligned}$$

Thus $P_{0j} = |C_j|$, so

$$\chi_0(g_j) = \frac{\chi_0(1)}{|C_j|}P_{0j} = 1.$$

The irreducible character χ_0 is called the trivial character.

The character table

Given a finite group, we have

$$\begin{aligned}\text{conjugacy classes: } & C_0 = \{1\}, C_1, \ldots, C_d, \\ \text{irreducible characters: } & \chi_0, \chi_1, \ldots, \chi_d.\end{aligned}$$

The character table T of G is the matrix whose (i,j)-entry is $\chi_i(g_j)$, where $g_j \in C_j$. Since

$$P_{ij} = \frac{|C_j|}{\chi_i(1)} \chi_i(g_j),$$

$$P = \begin{bmatrix} \ddots & & \\ & \chi_i(1) & \\ & & \ddots \end{bmatrix}^{-1} T \begin{bmatrix} \ddots & & \\ & |C_j| & \\ & & \ddots \end{bmatrix}.$$

This is why the first eigenmatrix P is sometimes called the character table of a commutative association scheme.

$P_{ij} = \frac{|C_j|}{\chi_i(1)}\chi_i(g_j) \rightsquigarrow Q_{ij} \rightsquigarrow E_j$

$$Q_{ij} = \frac{\operatorname{rank} E_j}{k_i}\overline{P_{ji}} = \frac{m_j^2}{|C_i|} \cdot \frac{|C_i|}{\chi_j(1)}\overline{\chi_j(g_i)} = \chi_j(1)\chi_j(g_i),$$

The central primitive idempotents of the group algebra is

$$\begin{aligned} E_j &= \frac{1}{|G|}\sum_{i=0}^{d} Q_{ij}A_i = \frac{\chi_j(1)}{|G|}\sum_{i=0}^{d}\overline{\chi_j(g_i)}A_i \\ &= \frac{\chi_j(1)}{|G|}\sum_{i=0}^{d}\overline{\chi_j(g_i)}\sum_{g\in C_i} P_g \\ &= \frac{\chi_j(1)}{|G|}\sum_{g\in G}\overline{\chi_j(g)}P_g. \end{aligned}$$

Summary

$$\begin{array}{ccc} \text{finite group} & \rightarrow & \text{character table } T \\ \downarrow & & \downarrow \text{normalize} \\ Z(\mathbb{C}[G]) = \mathbb{C}[\mathcal{R}] & \rightarrow & \text{eigenmatrix } P \end{array}$$

The above "finite group" is actually the right regular representation.

$$\begin{array}{ccc} \begin{array}{l}\text{multiplicity-free} \\ \text{finite group action}\end{array} & \rightarrow & {\color{red}??} \\ \downarrow & & \downarrow \\ \text{commutative } \mathbb{C}[\mathcal{R}] & \rightarrow & \text{eigenmatrix } P \end{array}$$

More generally

$$\begin{array}{ccc} \text{finite group action} & \rightarrow & {\color{red}??} \\ \downarrow & & \downarrow \\ \mathbb{C}[\mathcal{R}] & \rightarrow & {\color{red}??} \end{array}$$