Graph coverings 5.
Lifting automorphism problem

R. Nedela

Faculty of Applied Sciences, University of West Bohemia, Pilsen

Lifting Automorphism Problem

Let $p : Y \to X$ be a covering between graphs

LAP: **Does there exists an automorphism $\tilde{f}$ of Y associated with an automorphism $f \in Aut(X)$ such that the diagram**

$$\begin{tikzcd} Y \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & Y \arrow[d, "p"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

commutes?
When $\tilde{f}$ is a *lift* of f, we say $\tilde{f}$ *projects* onto f.

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2. If a covering is regular, then the lift of a group $H \leq CT(p)$ along p is an extension of $CT(p)$ by H,
3. One can treat group extensions in a combinatorial way,
4. It produces algorithms for computations of group extensions of prescribed type.

Basic properties

Let $p : \tilde{X} \to X$ be a covering.

1. if f lifts and f has a finite order in $\mathrm{Aut}(X)$, so does f^{-1},
2. lift of an automorphism is an automorphism,
3. if $A \leq \mathrm{Aut}(X)$ then the lifts of all automorphisms in A form a group, the *lift* of A, denoted by $\tilde{A} \leq \mathrm{Aut}(\tilde{X})$,
4. the lift of the trivial group is the covering transformation group $\mathrm{CT}(p)$.
5. There is an associated group epimorphism $p_{\mathcal{A}} : \tilde{\mathcal{A}} \to \mathcal{A}$ with $\mathrm{CT}(p)$ as its kernel.
6. The set of all lifts $L(f)$ of a given $f \in \mathrm{Aut}(X)$ is a coset of $\mathrm{CT}(p)$ in $\tilde{\mathcal{A}}$.
7. The trivial group lifts to $CT(p)$.

Lifting Automorphism Problem

Lemma 1. Let $p : \tilde{X} \to X$ be a covering projection of connected graphs and let $f \in \mathrm{Aut}(X)$ have a lift. Then each $\tilde{f} \in L(f)$ is uniquely determined by the image of a single vertex (as well as by the image of a single dart) of $\tilde{X}$.

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Example: Group of fibre-preserving automorphisms may be much larger than the group of lifts.

$\tilde{X} = \mathrm{Cay}(\mathbb{Z}_n; \mathbb{Z}_n - \{0\}) \cong K_n$ then $p(j,i) = i$ is a covering $p : \tilde{X} \to X$ from K_n onto a monopole. Set $m = \lfloor \frac{n-1}{2} \rfloor$ and $H \leq \mathrm{Aut}(X)$ be the maximal group of automorphism which lifts. Then

$$|\tilde{H}| \leq |\mathrm{CT}(p)| \cdot |H| = n2^m \cdot m! < n!$$

$\tilde{H}$ is the Frobenius group isomorphic to $Z_n \rtimes Z_n^*$.

Canonical double covering

Let $X = (D, V; I, \lambda)$ be a graph.
Its *canonical double covering* is defined by setting $\tilde{D} = D \times \mathbb{Z}_2$, $\tilde{V} = V \times \mathbb{Z}_2$, $\tilde{I}(x_i) = (I(x))_i$ and $\tilde{\lambda}(x_i) = \lambda(x)_{i+1}$.

1. $x_i \mapsto x$ is a (regular) 2-fold covering $p : \tilde{X} \to X$,
2. $\beta : x_i \mapsto x_{i+1}$ is a nontrivial element of $\mathrm{CT}(p) \cong Z_2$,
3. every automorphism $\psi \in \mathrm{Aut}(X)$ lifts to two automorphisms $\tilde{\psi}_j$, $j \in Z_2$, defined by $\tilde{\psi}_j(x_i) = \psi(x)_{j+i}$, for $i \in Z_2$,
4. $\mathrm{Aut}(X)$ lifts to $\tilde{\mathrm{Aut}}(X) \cong \mathrm{Aut}(X) \times Z_2$.

STABILITY PROBLEM!

Group of Lifts

LEMMA. Let $p : \tilde{X} \to X$ be a regular covering between connected graphs and let $\mathcal{A} = \mathrm{Aut}(X)$. Then the normalizer $N_{\mathrm{Aut}(\tilde{X})}(\mathrm{CT}(p))$ of $\mathrm{CT}(p)$ projects. In particular, the normalizer $N_{\mathrm{Aut}(\tilde{X})}(\mathrm{CT}(p))$ coincides with the group $\widetilde{\mathcal{A}}$ of all lifts.

Corollary

If A lifts along a regular covering projection p then it lifts to an extension of $\mathrm{CT}(p)$ by A.
In particular, let $p : \tilde{X} \to X$ be a regular covering projection of graphs. Let $A \leq \mathrm{Aut}(X)_v$ be a subgroup of the stabilizer of a vertex v. If A lifts then its lifting is a split extension of $\mathrm{CT}(p)$.

EXAMPLE.

- Represent the vertices of $K_{n,n}$, $n > 2$, and as $Z_n \times Z_2$ so that $V = \{(j,0) \mid j \in Z_n\} \cup \{(j,1) \mid j \in Z_n\}$ is the bipartition of $K_{n,n}$
- decompose the edges into n prefect matchings $M_0, M_1, \ldots, M_{n-1}$, where M_i is formed by the n edges joining a vertex $(j,0)$ to a vertex $(j+i,1)$ for $i,j \in Z_n$.
- arcs of $K_{n,n}$ are then triples (k,j,i), where (k,j) is a vertex and i is the colour of a matching,
- $p:(k,j,i) \to (j,i)$ is a covering from $K_{n,n}$ onto an n-valent dipole $\mathcal{D}(n)$ with $\mathrm{CT}(p)$ generated by $(j,0) \mapsto (j+1,0)$ and $(j,1) \mapsto (j+1,1)$ which is isomorphic to Z_n.

$G = \mathrm{Aut}(\mathcal{D}(n))$ is isomorphic $S_n \times Z_2$ whose order is $2 \cdot n!$, hence

$$|Lift(\mathrm{Aut}(X))| = |N_{\mathrm{Aut}(\tilde{X})}(\mathrm{CT}(p))| = n|\mathrm{Aut}(X)| = 2n \cdot n!.$$

The subgroup preserving vertex-fibres is the whole $Aut(K_{n,n})$ of size $2n!n!$.

Lifting Problem - Classical Approach

Theorem

Let $p : \tilde{X} \to X$ be a covering projection of connected graphs and let $f \in \mathrm{Aut}(X)$. Then f lifts to an $\tilde{f} \in \mathrm{Aut}(\tilde{X})$ if and only if, for an arbitrarily chosen base vertex $b \in X$, there exists a bijection $\phi : \mathrm{fib}_b \to \mathrm{fib}_{fb}$ so that the pair

$$(\phi, f_*) : (\mathrm{fib}_b, \pi^b) \to (\mathrm{fib}_{fb}, \pi^{fb})$$

becomes an **isomorphism of the fundamental group acting spaces** *and $\tilde{f}|_{\mathrm{fib}_b} = \phi$.*
Moreover, there is a one-to-one correspondence $\tilde{f} \leftrightarrow \phi$ between Lifts(f) and all functions ϕ for which (ϕ, f_*) is such an isomorphism, with the relation

$$\tilde{f}(\tilde{u}) = \phi(\tilde{u} \cdot W) \cdot fW^{-1} \text{ for any } W : p(\tilde{u}) \to b \text{ in } X.$$

That is, $\tilde{f}(\tilde{u})$ is the terminal vertex of the lifting of the walk fW^{-1} which is initiated at vertex $\phi(\tilde{u} \cdot W)$.

Lifting problem - Classical approach

Theorem

Let $p : \tilde{X} \to X$ be a covering projection of connected graphs, and let $f \in \mathrm{Aut}(X)$ and b be a vertex of X. Then there exists an isomorphism of actions

$$(\phi, f_*) : (\mathrm{fib}_b, \pi^b) \to (\mathrm{fib}_{fb}, \pi^{fb})$$

if and only if f_* maps the stabilizer $\pi_{\tilde{b}}$ of an arbitrarily chosen base point $\tilde{b} \in \mathrm{fib}_b$ isomorphically onto some stabilizer $\pi_{\tilde{v}} \leq \pi^{fb}$.
In this case we have $\tilde{v} = \phi\tilde{b}$. It gives a one-to-one correspondence $\tilde{v} \leftrightarrow \phi$ between the set $\{\tilde{v} \in \mathrm{fib}_{fb} \mid f_*\pi_{\tilde{b}} = \pi_{\tilde{v}}\}$ and all such admissible isomorphisms ϕ. In particular, the number of all liftings of f equals the cardinality $|\{\tilde{v} \in \mathrm{fib}_{fb} \mid f_*\pi_{\tilde{b}} = \pi_{\tilde{v}}\}|$.

Regular Coverings and Lifting Problem

There exists a covering transformation mapping $\tilde{u}_1 \in \mathrm{fib}_u$ to $\tilde{u}_2 \in \mathrm{fib}_u$ if and only if the stabilizers coincide: $\pi_{\tilde{u}_1} = \pi_{\tilde{u}_2}$.

Corollary

A covering projection of connected graphs $p : \tilde{X} \to X$ is regular if and only if, for an arbitrarily chosen vertex b of X, all the stabilizers under the action of π^b on fib_b coincide, that is to say, the stabilizer is a normal subgroup.

Lifting problem in terms of voltages

Let A be a group of automorphisms of X. We say that the voltage space $(F, G; \xi)$ is **locally A-invariant** at a vertex v if, for every $f \in A$ and for every walk $W \in \pi^v$, we have

$$\xi_W = 1 \implies \xi_{fW} = 1. \tag{1}$$

Theorem

Let $(F, G; \xi)$ be a monodromy voltage space associated with a covering $p : \tilde{X} \to X$ of connected graphs, and let f be an automorphism of X. If f has a lift then the voltage space $(F, G; \xi)$ is locally f-invariant,

Monodromy voltage space $=$, the respective action on a fibre is faithful, true for Cayley and Permutation voltage spaces.

Is the local invariance sufficient?

The local A-invariance is closely related to the requirement that for each $f \in A$ there exists an induced isomorphism $f^u_\# : G^u \to G^{fu}$ of local voltage groups such that the diagram

$$\begin{tikzcd}
\pi^u \arrow[r, "f^u_*"] \arrow[d, "\xi"'] & \pi^{fu} \arrow[d, "\xi"] \\
G^u \arrow[r, "f^u_\#"] & G^{fu}
\end{tikzcd}$$

commutates; in other words, $f^u_\#(\xi_W) = \xi_{f^u W}$ for any $W \in \pi^u$.

Characterization in terms of voltages

Theorem

Let $(F, G; \xi)$ be a monodromy voltage space associated with a covering $p : \tilde{X} \to X$ of connected graphs, and let f be an automorphism of X. Then the following statements are equivalent:

(1) *f has a lift,*

(2) *the mapping $f_{\#}^{b} : G^{b} \to G^{fb}$ taking $\xi_W \mapsto \xi_{fW}$ is a well-defined isomorphism of the local voltage groups as acting groups on F.*

Ex. Local invariance does not imply existence of lift even if the voltage space is a monodromy space.

Lifting problem in terms of voltages - regular case

Corollary

Let the covering as well as the voltage space be regular. Then the following three statements are equivalent:

(1) f *has a lift,*

(2) *the voltage space* $(F, G; \xi)$ *is locally* f*-invariant,*

(3) *the mapping* $f_\#^b : G^b \to G^{fb}$ *taking* $\xi_W \mapsto \xi_{fW}$ *is a well-defined isomorphism of the local voltage groups with respective actions on* F.

Moreover, if (G, G, ξ) *is a Cayley voltage space then* $G^b = G = G^{fb}$ *and* $f_\#^b$ *is an automorphism of* G.

Lifting problem - solution in terms of equations in $Sym(n)$

Corollary

Let $(F, Sym_R\, F; \xi)$ be a locally transitive permutation voltage space on a graph X. Then, an automorphism f of X has a lift if and only if the system of equations

$$\xi_S \cdot \tau = \tau \cdot \xi_{fS}$$

in $Sym_R\, F$ has a solution τ, where S runs through a generating set for π^b. Moreover, there is a one-to-one correspondence between all solutions and the lifts of f.

Irregular covers of K_n on which $\mathrm{Aut}(K_n)$ lifts

1. Set $V = \{0, 1, 2, \ldots, n-1\}$ of K_n
2. Choose the spanning tree T of K_n to be the star containing edges $\{(0, j), (j, 0)\}$ for $j = 1, 2, \ldots, n-1$
3. a T-reduced permutation voltage space (F, H, ξ) on X, where $F = Z_n - \{0\}$ and $H \subseteq Sym(F)$
4. by setting $\xi_{(i,j)} = (i, j)$ for each cotree dart (i, j).

Irregular covers of K_n on which $\mathrm{Aut}(K_n)$ lifts

The system of equations for (i,j), $i \neq 0$, $j \neq 0$

$$\begin{array}{rclll}
(i,j)\cdot\tau_{(i,j)} & = & \tau_{(i,j)}\cdot(i,j) & & \\
(i,k)\cdot\tau_{(i,j)} & = & \tau_{(i,j)}\cdot(j,k) & \text{for all} & k \neq 0 \\
(j,k)\cdot\tau_{(i,j)} & = & \tau_{(i,j)}\cdot(i,k) & \text{for all} & k \neq 0 \\
(k,\ell)\cdot\tau_{(i,j)} & = & \tau_{(i,j)}\cdot(k,\ell) & \text{for all} & k,\ell \neq 0
\end{array}$$

has a solution $\tau_{(i,j)} = (i,j)$.

Similarly for an automorphism $(0,1) \in \mathrm{Aut}(X)$ the associated system of equations

$$\begin{array}{rcll} (k,1)\cdot\tau_{(0,1)} & = & \tau_{(i,j)}\cdot(k,1) & \text{for all} \quad k \neq 0,1 \\ (i,j)\cdot\tau_{(0,1)} & = & \tau_{(0,1)}\cdot(i,j) & \text{for all} \quad i,j \neq 0,1 \end{array}$$

has a trivial solution $\tau_{(0,1)} = 1$. Hence the automorphism group of K_n lifts.

Example: Automorphisms lift along the homology coverings

Let $D = \{x_i^{\pm} \mid i = 1, 2, \ldots, d\}$ be the set of all darts and let $F_d = (Z_n, +)^d$ be the free product of cyclic groups Z_n of order n, for some $n \geq 2$ with a generating set $a_1, a_2, \ldots, a_d$.

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Set $\xi(x_i) = a_i$.

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Then each automorphism of X lifts!!! Why?
The covering graph is disconnected, it consists of isomorphic components of size $|X| \cdot n^{\beta}$, where β is the Betti number.

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An application: **There are infinitely many 5-transitive cubic graphs.**