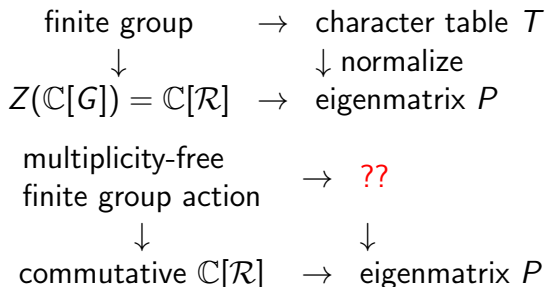


Permutation representations of finite groups and association schemes (Part 6)

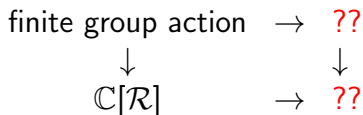
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Review



More generally



We will discuss “??” in this lecture.

The central primitive idempotents of the group algebra

We consider another transitive action $(G; \Omega)$, so we distinguish permutation matrices and idempotents as follows:

P_g^R for regular representation, P_g for $(G; \Omega)$,

E_j^R for regular representation, E_j for $(G; \Omega)$.

So the central primitive idempotents of the group algebra (right regular representation) are

$$E_j^R = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g^R.$$

Homomorphic image

$P_g^R \rightarrow P_g$ extends linearly to a homomorphism of algebras

$$\mathbb{C}[G] = \mathbb{C}[\{P_g^R \mid g \in G\}] \rightarrow \mathbb{C}[\{P_g \mid g \in G\}].$$

$$\left[\begin{array}{ccc} \ddots & & \\ & \ddots & \\ & & \boxed{I_{m_j} \otimes M_{m_j}(\mathbb{C})} & \\ & & & \ddots \\ & & & & \ddots \end{array} \right] \rightarrow \left[\begin{array}{ccc} \ddots & & \\ & \ddots & \\ & & \boxed{I_{n_j} \otimes M_{m_j}(\mathbb{C})} & \\ & & & \ddots \\ & & & & \ddots \end{array} \right]$$

for some n_j (possibly 0).

$$E_j^R = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g^R \mapsto E_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g.$$

Can we express E_j in terms of the adjacency matrices?

E_j in terms of A_i

Since $E_j \in \mathbb{C}[\mathcal{R}]$ (the adjacency algebra of the action $(G; \Omega)$), it is a linear combination of A_i 's:

$$E_j = \sum_{i=0}^d c_{ij} A_i = \frac{1}{|\Omega|} \sum_{i=0}^d Q_{ij} A_i \text{ (if commutative).}$$

What we have from general representation theory is:

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g.$$

Can we combine P_g to get A_i ? Yes, by Curtis–Fossum (1969):

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^d \frac{1}{|k_i|} \sum_{g \in H a_i H} \overline{\chi_j(g)} A_i$$

for some $a_0, \dots, a_d \in G$.

Double cosets

For the **transitive** action $(G; \Omega)$, fix $\omega \in \Omega$ and let $H = G_\omega$. Then we have a double coset decomposition

$$G = \bigcup_{i=0}^d H a_i H \quad (\text{disjoint, we may assume } a_0 = 1).$$

Then the 2-orbits are

$$R_i = (\omega, \omega^{a_i})^G = \{(\omega^x, \omega^{a_i x}) \mid x \in G\} \quad (i = 0, 1, \dots, d).$$

Indeed, for $(\alpha, \beta) \in \Omega^2$, $\exists g \in G$, $\exists h, h' \in H$, $\exists i$ such that

$$\alpha = \omega^g, \quad \beta^{g^{-1}} = \omega^{h' a_i h}.$$

Define $x = hg$ to verify $(\omega^x, \omega^{a_i x}) = (\omega^{hg}, \omega^{a_i hg}) = (\alpha, \beta)$.

Properties of R_i

$$R_i = (\omega, \omega^{a_i})^G = \{(\omega^x, \omega^{a_i x}) \mid x \in G\}.$$

For $g \in G$,

$$(\omega, \omega^g) \in R_i \iff g \in Ha_iH.$$

Indeed,

$$\begin{aligned}(\omega, \omega^g) \in R_i &\iff \exists x \in G, (\omega, \omega^g) = (\omega^x, \omega^{a_i x}) \\&\iff \exists x \in G, \omega = \omega^x, \omega^g = \omega^{a_i x} \\&\iff \exists x \in H, \omega^g = \omega^{a_i x} \\&\iff \exists x \in H, gx^{-1}a_i^{-1} \in H \\&\iff \exists x \in H, g \in Ha_i x \\&\iff g \in Ha_i H.\end{aligned}$$

Proof (I)

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g \stackrel{?}{=} \frac{\chi_j(1)}{|G|} \sum_{i=0}^d \frac{1}{|k_i|} \sum_{g \in Ha_i H} \overline{\chi_j(g)} A_i.$$

or equivalently,

$$\sum_{g \in G} \overline{\chi_j(g)} P_g \stackrel{?}{=} \sum_{i=0}^d \frac{1}{k_i} \sum_{g \in Ha_i H} \overline{\chi_j(g)} A_i.$$

Proof (II) $H = G_\omega$

$$\sum_{g \in G} \overline{\chi_j(g)} P_g \stackrel{?}{=} \sum_{i=0}^d \frac{1}{k_i} \sum_{g \in Ha_i H} \overline{\chi_j(g)} A_i.$$

For $(\beta, \gamma) \in R_i$, the (β, γ) -entry of the left-hand side $\in \mathbb{C}[\mathcal{R}]$ is

$$\begin{aligned} \frac{1}{k_i} \sum_{\substack{\alpha \in \Omega \\ (\omega, \alpha) \in R_i}} \sum_{g \in G} \overline{\chi_j(g)} (P_g)_{\omega, \alpha} &= \frac{1}{k_i} \sum_{\substack{\alpha \in \Omega \\ (\omega, \alpha) \in R_i}} \sum_{\substack{g \in G \\ \omega^g = \alpha}} \overline{\chi_j(g)} \\ &= \frac{1}{k_i} \sum_{\substack{g \in G \\ (\omega, \omega^g) \in R_i}} \overline{\chi_j(g)} \\ &= \frac{1}{k_i} \sum_{g \in Ha_i H} \overline{\chi_j(g)}, \end{aligned}$$

since $(\omega, \omega^g) \in R_i \iff g \in Ha_i H$.

The character values of a finite group

Recall

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^d \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i.$$

$\chi_j(g) = \text{tr } \Delta_j(P_g)$ is the sum of eigenvalues of $\Delta_j(P_g)$.

Since

$$\begin{aligned} \exists \ell, \forall g \in G, g^\ell = 1 &\implies P_g^\ell = I_\Omega \implies \Delta_j(P_g)^\ell = I_{m_j} \\ &\implies \text{eigenvalues of } \Delta_j(P_g) \text{ are roots of } 1, \end{aligned}$$

(the least such ℓ is called the **exponent** of G)

$\chi_j(g)$ is expressible in terms of $\zeta = \exp \frac{2\pi\sqrt{-1}}{\ell}$, that is, it belongs to the **cyclotomic field** $\mathbb{Q}(\zeta)$.

Characters and eigenvalues

From

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^d \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i,$$

we find the “character”, that is, the trace of A_i on one of the equal j -th block

$$\frac{1}{m_j} \operatorname{tr}(E_j A_i),$$

since

$$A_i \in U \begin{bmatrix} \ddots & & \\ & \boxed{M_{n_j}(\mathbb{C}) \otimes I_{m_j}} & \\ & & \ddots \end{bmatrix} U^* \ni E_j = U \begin{bmatrix} \ddots & & \\ & I_{n_j} \otimes I_{m_j} & \\ & & \ddots \end{bmatrix} U^*.$$

If multiplicity-free. . .

If multiplicity-free, then $n_j = 1$

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^d \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i.$$

The trace of A_i on one of the equal 1×1 j -th block is

$$\frac{1}{m_j} \operatorname{tr}(E_j A_i) \text{ which is an eigenvalue of } A_i,$$

since

$$A_i \in U \begin{bmatrix} \ddots & & \\ & \boxed{\mathbb{C}I_{m_j}} & \\ & & \ddots \end{bmatrix} U^* \ni E_j = U \begin{bmatrix} \ddots & & \\ & I_{m_j} & \\ & & \ddots \end{bmatrix} U^*.$$

Eigenvalues of A_i belong to the cyclotomic field $\mathbb{Q}(\zeta)$.

Summary

For a multiplicity-free permutation representation $(G; \Omega)$, the eigenvalues of the adjacency matrices belong to a **cyclotomic field** $\mathbb{Q}(\zeta)$, where ζ is a primitive ℓ -th root of unity, and ℓ is the exponent of G .

This consequence strongly depends on the character theory of finite groups, so it does **not** apply to general association schemes.

However, it is **unknown** whether there exists a commutative association scheme whose eigenvalues do **not** belong to a cyclotomic field.

In the next lecture, we prove that the entries of the eigenmatrix of a commutative association scheme **do** belong to a cyclotomic field, provided its Krein parameters are rational (Munemasa, 1991).