Permutation representations of finite groups and association schemes (Part 6)

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Review

$$
\begin{array}{ccl}
\text{finite group} & \to & \text{character table } T \\
\downarrow & & \downarrow \text{normalize} \\
Z(\mathbb{C}[G]) = \mathbb{C}[\mathcal{R}] & \to & \text{eigenmatrix } P
\end{array}
$$

$$
\begin{array}{ccl}
\begin{array}{l}\text{multiplicity-free} \\ \text{finite group action}\end{array} & \to & {\color{red}??} \\
\downarrow & & \downarrow \\
\text{commutative } \mathbb{C}[\mathcal{R}] & \to & \text{eigenmatrix } P
\end{array}
$$

More generally

$$
\begin{array}{ccc}
\text{finite group action} & \to & {\color{red}??} \\
\downarrow & & \downarrow \\
\mathbb{C}[\mathcal{R}] & \to & {\color{red}??}
\end{array}
$$

We will discuss “??” in this lecture.

The central primitive idempotents of the group algebra

We consider another transitive action $(G;\Omega)$, so we distinguish permutation matrices and idempotents as follows:

$$P_g^R \text{ for regular representation, } P_g \text{ for } (G;\Omega),$$

$$E_j^R \text{ for regular representation, } E_j \text{ for } (G;\Omega).$$

So the central primitive idempotents of the group algebra (right regular representation) are

$$E_j^R = \frac{\chi_j(1)}{|G|}\sum_{g\in G}\overline{\chi_j(g)}P_g^R.$$

Homomorphic image

$P_g^R \to P_g$ extends linearly to a homomorphism of algebras

$$\mathbb{C}[G] = \mathbb{C}[\{P_g^R \mid g \in G\}] \to \mathbb{C}[\{P_g \mid g \in G\}].$$

$$\begin{bmatrix} \ddots & & & \\ & \ddots & & & \\ & & \boxed{I_{m_j} \otimes M_{m_j}(\mathbb{C})} & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix} \to \begin{bmatrix} \ddots & & \\ & \boxed{I_{n_j} \otimes M_{m_j}(\mathbb{C})} & \\ & & \ddots \end{bmatrix}$$

for some n_j (possibly 0).

$$E_j^R = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g^R \mapsto E_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g.$$

Can we express E_j in terms of the adjacency matrices?

E_j in terms of A_i

Since $E_j \in \mathbb{C}[\mathcal{R}]$ (the adjacency algebra of the action $(G;\Omega)$, it is a linear combination of A_i's:

$$E_j = \sum_{i=0}^{d} c_{ij} A_i \quad = \frac{1}{|\Omega|} \sum_{i=0}^{d} Q_{ij} A_i \text{ (if commutative)}.$$

What we have from general representation theory is:

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g.$$

Can we combine P_g to get A_i? Yes, by Curtis–Fossum (1969):

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^{d} \frac{1}{|k_i|} \sum_{g \in H a_i H} \overline{\chi_j(g)} A_i$$

for some $a_0, \ldots, a_d \in G$.

Double cosets

For the transitive action $(G;\Omega)$, fix $\omega \in \Omega$ and let $H = G_\omega$. Then we have a double coset decomposition

$$G = \bigcup_{i=0}^{d} Ha_iH \quad \text{(disjoint, we may assume } a_0 = 1\text{)}.$$

Then the 2-orbits are

$$R_i = (\omega, \omega^{a_i})^G = \{(\omega^x, \omega^{a_i x}) \mid x \in G\} \quad (i = 0, 1, \ldots, d).$$

Indeed, for $(\alpha, \beta) \in \Omega^2$, $\exists g \in G$, $\exists h, h' \in H$, $\exists i$ such that

$$\alpha = \omega^g, \quad \beta^{g^{-1}} = \omega^{h' a_i h}.$$

Define $x = hg$ to verify $(\omega^x, \omega^{a_i x}) = (\omega^{hg}, \omega^{a_i hg}) = (\alpha, \beta)$.

Properties of R_i

$$R_i = (\omega, \omega^{a_i})^G = \{(\omega^x, \omega^{a_i x}) \mid x \in G\}.$$

For $g \in G$,

$$(\omega, \omega^g) \in R_i \iff g \in Ha_iH.$$

Indeed,

$$\begin{aligned}
(\omega, \omega^g) \in R_i &\iff \exists x \in G,\ (\omega, \omega^g) = (\omega^x, \omega^{a_i x}) \\
&\iff \exists x \in G,\ \omega = \omega^x,\ \omega^g = \omega^{a_i x} \\
&\iff \exists x \in H,\ \omega^g = \omega^{a_i x} \\
&\iff \exists x \in H,\ gx^{-1}a_i^{-1} \in H \\
&\iff \exists x \in H,\ g \in Ha_i x \\
&\iff g \in Ha_iH.
\end{aligned}$$

Proof (I)

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g \stackrel{?}{=} \frac{\chi_j(1)}{|G|} \sum_{i=0}^{d} \frac{1}{|k_i|} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i.$$

or equivalently,

$$\sum_{g \in G} \overline{\chi_j(g)} P_g \stackrel{?}{=} \sum_{i=0}^{d} \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i.$$

Proof (II) $H = G_\omega$

$$\sum_{g\in G} \overline{\chi_j(g)} P_g \stackrel{?}{=} \sum_{i=0}^{d} \frac{1}{k_i} \sum_{g\in Ha_iH} \overline{\chi_j(g)} A_i.$$

For $(\beta,\gamma) \in R_i$, the (β,γ)-entry of the left-hand side $\in \mathbb{C}[\mathcal{R}]$ is

$$\begin{aligned}
\frac{1}{k_i} \sum_{\substack{\alpha\in\Omega \\ (\omega,\alpha)\in R_i}} \sum_{g\in G} \overline{\chi_j(g)} (P_g)_{\omega,\alpha} &= \frac{1}{k_i} \sum_{\substack{\alpha\in\Omega \\ (\omega,\alpha)\in R_i}} \sum_{\substack{g\in G \\ \omega^g=\alpha}} \overline{\chi_j(g)} \\
&= \frac{1}{k_i} \sum_{\substack{g\in G \\ (\omega,\omega^g)\in R_i}} \overline{\chi_j(g)} \\
&= \frac{1}{k_i} \sum_{g\in Ha_iH} \overline{\chi_j(g)},
\end{aligned}$$

since $(\omega,\omega^g) \in R_i \iff g \in Ha_iH$.

The character values of a finite group

Recall

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^{d} \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i.$$

$$\chi_j(g) = \operatorname{tr} \Delta_j(P_g) \text{ is the sum of eigenvalues of } \Delta_j(P_g).$$

Since

$$\begin{aligned} \exists \ell,\ \forall g \in G,\ g^\ell = 1 &\implies P_g^\ell = I_\Omega \implies \Delta_j(P_g)^\ell = I_{m_j} \\ &\implies \text{eigenvalues of } \Delta_j(P_g) \text{ are roots of } 1, \end{aligned}$$

(the least such ℓ is called the exponent of G)

$\chi_j(g)$ is expressible in terms of $\zeta = \exp \frac{2\pi\sqrt{-1}}{\ell}$, that is, it belongs to the cyclotomic field $\mathbb{Q}(\zeta)$.

Characters and eigenvalues

From

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^{d} \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i,$$

we find the "character", that is, the trace of A_i on one of the equal j-th block

$$\frac{1}{m_j} \operatorname{tr}(E_j A_i),$$

since

$$A_i \in U \begin{bmatrix} \ddots & & \\ & \boxed{M_{n_j}(\mathbb{C}) \otimes I_{m_j}} & \\ & & \ddots \end{bmatrix} U^* \ni E_j = U \begin{bmatrix} \ddots & & \\ & I_{n_j} \otimes I_{m_j} & \\ & & \ddots \end{bmatrix} U^*.$$

If multiplicity-free...

If multiplicity-free, then $n_j = 1$

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^{d} \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i.$$

The trace of A_i on one of the equal 1×1 j-th block is

$$\frac{1}{m_j} \mathrm{tr}(E_j A_i) \text{ which is an eigenvalue of } A_i,$$

since

$$A_i \in U \begin{bmatrix} \ddots & & \\ & \boxed{\mathbb{C} I_{m_j}} & \\ & & \ddots \end{bmatrix} U^* \ni E_j = U \begin{bmatrix} \ddots & & \\ & I_{m_j} & \\ & & \ddots \end{bmatrix} U^*.$$

Eigenvalues of A_i belong to the cyclotomic field $\mathbb{Q}(\zeta)$.

Summary

For a multiplicity-free permutation representation $(G; \Omega)$, the eigenvalues of the adjacency matrices belong to a cyclotomic field $\mathbb{Q}(\zeta)$, where ζ is a primitive ℓ-th root of unity, and ℓ is the exponent of G.

This consequence strongly depends on the character theory of finite groups, so it does not apply to general association schemes.

However, it is unknown whether there exists a commutative association scheme whose eigenvalues do not belong to a cyclotomic field.

In the next lecture, we prove that the entries of the eigenmatrix of a commutative association scheme do belong to a cyclotomic field, provided its Krein parameters are rational (Munemasa, 1991).