Graph coverings 6.
Lifting automorphism problem, Abelian $CT(p)$

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Lifting problem with abelian $CT(p)$

For simplicity we consider graphs without loops and semiedges.
A matrix is called **unimodular**, if its determinant is ± 1. We establish a homomorphism of $Aut(X) \to U_\beta$ into the group of unimodular $(\beta \times \beta)$-matrices taking $f \to M(f)$ as follows:

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Fix a linear order of the cotree darts with a chosen orientation $(x_1, x_2, \ldots, x_\beta)$ with each x_j we associate j-th collumn of M,

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Form a matrix $M(f) = M_T(f) = (m_{i,j})_{\beta \times \beta}$ by setting
$m_{i,j} = 1$ if $x_j \in Row(i)$,
$m_{i,j} = -1$ if $x_j^{-1} \in Row(i)$, and
$m_{i,j} = 0$ otherwise.

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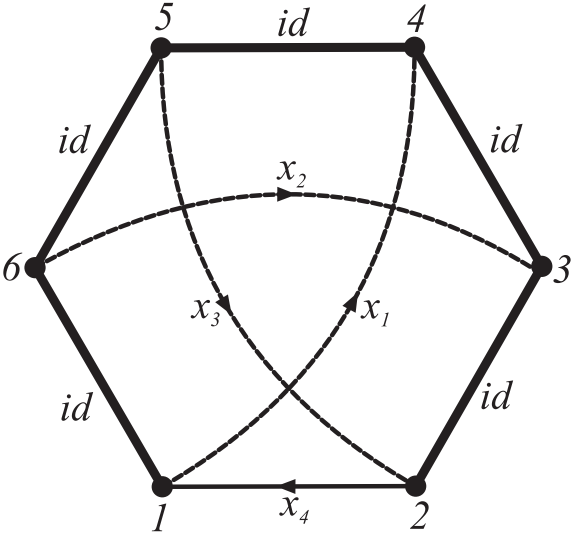

Figure: $K_{3,3}$ with T-reduced voltages

Example

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$r(C_1) = (2561)$, $r(C_2) = (1456)$, $r(C_3) = (6345)$, $r(C_4) = C_4$, and
$\ell(C_1) = (1652)$, $\ell(C_2) = (2365)$, $\ell(C_3) = (5436)$, $\ell(C_4) = (412563)$. By definition we have

$$M_T(r) = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_T(\ell) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \end{bmatrix}.$$

Why $f \mapsto M_T(f)$ is important in the lifting problem?

If $\xi : \pi(X, x_0) \to A$ is a Cayley voltage assignment taking values in an **abelian** group A, then

$$\xi(S_i) = W_i^{-1} C_i W_i = \xi(W_i)^{-1} \xi(C_i) \xi(W_i) = \xi(C_i)$$

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We write a T-reduced ξ as a column vector $\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_\beta)^t$, where $\xi_i = \xi(x_i)$, x_i is a cotree dart.

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Recall: $f \in \mathrm{Aut}(f)$ lifts if and only if $f_\# : A \to A$ is a group automorphism, and $f_\#(g) = f_\#(\xi_W) = \xi_{f(W)}$, for some $W \in \pi(X, x_0)$. However, $M_T(f)\vec{\xi} = (f_\#(\xi_1), f_\#(\xi_2), \ldots, f_\#(\xi_\beta))^t$, thus $M_T(f)$ determines the group automorphism $f_\# : A \to A$, if exists!

Solution of the abelian and cyclic case

Theorem

Let X be a connected graph with cycle rank β and a spanning tree T and let $(\mathcal{A}, \mathcal{A}, \xi)$ be a T-reduced locally transitive Cayley voltage space on X with an abelian group $\mathcal{A}$. Then the following statements are equivalent.

(1) *An automorphism f of X lifts,*
(2) *$(\mathcal{A}, \mathcal{A}, \xi)$ is locally f-invariant,*
(3) *$\vec{\xi}^{\perp} = (M\vec{\xi})^{\perp}$, that is, the orthogonal complements of the vectors $\vec{\xi}$ and $M\vec{\xi}$ are identical, where $M = M_T(f)$ is the matrix representation of f with respect to the spanning tree T.*

Moreover, if $\mathcal{A} = \mathbb{Z}_n$ is cyclic then the above statements are equivalent to

(4) *there exists an eigenvalue α coprime to n such that $M\vec{\xi} = \alpha\vec{\xi}$.*

Recall:

$$\vec{\xi}^{\perp} = \langle\vec{\xi}\rangle^{\perp} = \{\vec{a} \mid \vec{a} \cdot \vec{\xi} = 0\}.$$

Elementary abelian case

Recall: A is elementary abelian if it is a product of cyclic groups of order p, for some prime p, it is convenient to view A as the β-dimensional vector space over the field of charactersitic p. Now the statement $(M_T(f)(\vec{\xi}))^{\perp} = \vec{\xi}^{\perp}$ converts to the statement that

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The vector spaces of the solutions of two associated systems of linear equations $\sum_{i=1}^{\beta} x_i\vec{\xi_i} = \vec{0}$ and $\sum_{i=1}^{\beta} x_i M\vec{\xi_i} = \vec{0}$ in variables x_i is the same.

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Theorem

f lifts along an elementary abelian covering given by ξ if and only if the collumn space of $[\vec{\xi}]$ is $M(f)$-invariant.

Elementary abelian case, an algorithm

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(3) For each M-invariant subspace W, choose a basis $\alpha = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ with $\mathbf{a}_i \in \mathbb{Z}_p^\beta$ for the subspace W.

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(4) Now, define the voltages on the cotree darts by the rows of the matrix $\begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix}_{\beta \times n}$.

Which coverings determined by $\vec{\xi}$ are isomorphic?

Theorem

Let X be a graph and let M be a $\beta \times \beta$ matrix representing an automorphism of X. Given an M-invariant subspace W (with respect to left action), the covering graph projection derived from the Cayley voltage assignment (Z_p^n, Z_p^n, ξ), where $n = \dim W$, does not depend on the choice of the basis for W.

If W_1 and W_2 are two such subspaces, then the respective covering projections $\tilde{X}_i \to X$ are isomorphic if and only if there is an automorphism g of X such that $W_2 = M(g)W_1$.

How to determine M-invariant subspaces?

Theorem

(Maschke) *Let $G \leq \mathrm{GL}(n, \mathbb{F})$ be a group of linear transformations over a finite field $\mathbb{F}$ of characteristic p and let $\gcd(|G|, p) = 1$. Then the space $\mathbb{F}^n$ decomposes uniquely into a direct sum of minimal G-invariant subspaces and every G-invariant subspace is a direct sum the minimal ones.*

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If we want to lift a group of automorphisms generated by $f_1, f_2, \ldots, f_r$ automorphisms we need to compute the intersection of the respective $M(f_i)$-invariant subspaces.