Permutation representations of finite groups and association schemes (Part 7)

Akihiro Munemasa

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Contents

Review

In a commutative association scheme $\mathcal{R}$,

$$(A_0, \ldots, A_d) = (E_0, \ldots, E_d)P,$$
$$(E_0, \ldots, E_d) = \frac{1}{|\Omega|}(A_0, \ldots, A_d)Q,$$
$$E_i \circ E_j = \frac{1}{|\Omega|}\sum_{k=0}^{d} q_{ij}^k E_k.$$

q_{ij}^k are called Krein parameters.

The right regular representation

The central primitive idempotents are

$$\begin{aligned} E_j &= \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g \\ &= U \begin{bmatrix} 0 & & \\ & \boxed{I_{m_j^2}} & \\ & & 0 \end{bmatrix} U^* \quad (m_j = \chi_j(1)). \end{aligned}$$

Thus

$$E_i E_j = \delta_{ij} E_i,$$

$$E_j = E_j^* = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \chi_j(g) P_{g^{-1}} \implies \overline{\chi_j(g)} = \chi_j(g^{-1}).$$

Orthogonality relations

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g, \quad \overline{\chi_j(g)} = \chi_j(g^{-1})$$

$$E_i E_j = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \overline{\chi_i(g)} P_g \frac{\chi_j(1)}{|G|} \sum_{h \in G} \overline{\chi_j(h)} P_h,$$

$$= \frac{\chi_i(1)\chi_j(1)}{|G|^2} \sum_{g,h \in G} \overline{\chi_i(g)\chi_j(h)} P_{gh}.$$

If $i \neq j$, then the coefficient of P_1 is

$$0 = \sum_{h \in G} \overline{\chi_i(h^{-1})\chi_j(h)} = \sum_{h \in G} \chi_i(h)\overline{\chi_j(h)}.$$

Orthogonality relations $\sum_{h\in G}\chi_i(h)\overline{\chi_j(h)} = 0$.

$$E_j = \frac{\chi_j(1)}{|G|}\sum_{g\in G}\overline{\chi_j(g)}P_g, \quad \overline{\chi_j(g)} = \chi_j(g^{-1})$$

$$E_j = {E_j}^2 = \frac{\chi_j(1)^2}{|G|^2}\sum_{g,h\in G}\overline{\chi_j(g)\chi_j(h)}P_{gh}$$

The coefficient of P_1 is

$$\frac{\chi_j(1)}{|G|}\overline{\chi_j(1)} = \frac{\chi_j(1)^2}{|G|^2}\sum_{h\in G}\chi_j(h)\overline{\chi_j(h)}.$$

$$1 = \frac{1}{|G|}\sum_{h\in G}\chi_j(h)\overline{\chi_j(h)}.$$

Orthonormal basis of class functions

$$\frac{1}{|G|}\sum_{h\in G}\chi_i(h)\overline{\chi_j(h)} = \delta_{ij}.$$

For two functions $\chi, \psi : G \to \mathbb{C}$, we write

$$(\chi, \psi)_G = \frac{1}{|G|}\sum_{h\in G}\chi(h)\overline{\psi(h)}.$$

so

$$(\chi_i, \chi_j)_G = \delta_{ij}.$$

Theorem

If G has r conjugacy classes, then there are r irreducible characters, and they form an orthonormal basis of class functions.

Krein parameters

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \overline{\chi_j(g)} P_g$$

$$E_i \circ E_j = \left(\frac{\chi_i(1)}{|G|} \sum_{g \in G} \overline{\chi_i(g)} P_g \right) \circ \left(\frac{\chi_j(1)}{|G|} \sum_{h \in G} \overline{\chi_j(h)} P_h \right)$$

$$\frac{1}{|G|} \sum_{k=0}^{d} q_{ij}^k E_k = \frac{\chi_i(1)\chi_j(1)}{|G|^2} \sum_{h \in G} \overline{\chi_i(h)\chi_j(h)} P_h$$

$$q_{ij}^k E_k = \frac{\chi_i(1)\chi_j(1)}{|G|} \sum_{h \in G} \overline{\chi_i(h)\chi_j(h)} P_h E_k$$

This leads to

$$q_{ij}^k = \frac{\chi_i(1)\chi_j(1)}{\chi_k(1)} (\chi_i \chi_j, \chi_k)_G$$

Tensor product coefficients

$$(\chi_i \chi_j, \chi_k)_G$$

is called a tensor product coefficient. Recall

$$\chi_i(g) = \operatorname{tr} \phi_i(g), \text{ where } \phi_i : G \to GL_{m_i}(\mathbb{C})$$

$$\chi_i(g)\chi_j(g) = \operatorname{tr}(\phi_i(g) \otimes \phi_j(g)).$$

$$\phi_i \otimes \phi_j : G \to GL_{m_i m_j}(\mathbb{C})$$

is called the tensor product of the representations ϕ_i, ϕ_j. This is again a representation. The tensor product coefficient counts the multiplicity $m = (\chi_i \chi_j, \chi_k)_G$:

$$\phi_i \otimes \phi_j = \begin{bmatrix} \ddots & & \\ & \boxed{\phi_k \otimes I_m} & \\ & & \ddots \end{bmatrix}$$

q_{ij}^k in commutative association schemes

In a commutative association scheme $\mathcal{R}$,

$$(A_0, \ldots, A_d) = (E_0, \ldots, E_d)P,$$

$$(E_0, \ldots, E_d) = \frac{1}{|\Omega|}(A_0, \ldots, A_d)Q,$$

$$E_i \circ E_j = \frac{1}{|\Omega|}\sum_{k=0}^{d} q_{ij}^k E_k.$$

What are the Krein parameters?

For a transitive action $(G;\Omega)$,

$$E_j = \frac{\chi_j(1)}{|G|}\sum_{i=0}^{d}\frac{1}{k_i}\sum_{g\in Ha_iH}\overline{\chi_j(g)}A_i.$$

Indices of the Krein parameters i, j, k still correspond to (some, but not all) irreducible characters.

Scott's theorem

For a transitive action $(G;\Omega)$,

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^{d} \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i.$$

Theorem (Scott (1977))

Let $(G;\Omega)$ be a transitive action, and suppose it is multiplicity-free. Then

$$q_{ij}^k \neq 0 \implies (\chi_i\chi_j, \chi_k) \neq 0.$$

Let $V_i = E_i\mathbb{C}^\Omega$. Then

$$\begin{aligned} V_i \otimes V_j &\supseteq \mathrm{Span}\{E_ie_\alpha \otimes E_je_\alpha \mid \alpha \in \Omega\} \\ &\to \mathrm{Span}\{E_k(E_i \circ E_j)e_\alpha \mid \alpha \in \Omega\} \subseteq V_k. \end{aligned}$$

In the final lecture

Note: $\mathrm{tr}(A_i A_j) = \delta_{ii'} k_i |\Omega|$
For a transitive action $(G;\Omega)$, we have central primitive idempotents in terms of adjacency matrices:

$$E_j = \frac{\chi_j(1)}{|G|} \sum_{i=0}^{d} \frac{1}{k_i} \sum_{g \in Ha_iH} \overline{\chi_j(g)} A_i$$

If the action is multiplicity-free, then the eigenvalues of A_i are

$$\frac{1}{\chi_j(1)} \mathrm{tr}(E_j A_i) = \frac{1}{|H|} \sum_{g \in Ha_iH} \chi_j(g) \quad (j = 0, 1, \ldots, d),$$

and since $\chi_j(g)$ is a sum of roots of 1, they belong to a cyclotomic field $\mathbb{Q}(\zeta)$, where

$$\zeta = \exp \frac{2\pi\sqrt{-1}}{\ell}, \quad \ell = \text{ the exponent of } G.$$

Open problem (Bannai–Ito, 1984)

Not all commutative association schemes are obtained as the set of 2-orbits of a multiplicity-free action.

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \quad (j = 0, 1, \ldots, d).$$

Eigenvalues of A_i were denoted by P_{ij}, the entries of the first eigenmatrix:

$$(A_0, \ldots, A_d) = (E_0, \ldots, E_d)P.$$

Open Problem

Given a commutative association scheme $\mathcal{R}$, do the entries of P belong to a cyclotomic field?

"Yes" if $\mathcal{R} = \Omega^2/G$ for some (multiplicity-free) permutation group.