Graphs Coverings 7
Harmonic morphism between graphs

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Generalisation to branched coverings

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f locally uniform if for every vertex v of $V(X)$, there exists a constant m_v such that for every ordinary dart $x \in D_{fv}$, we have $|D_v \cap f^{-1}(x)| = m_v$, ($D_v$ is the set of darts based a v).

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f harmonic if it is a **locally uniform epimorphism between connected** graphs

If Y has no semiedges, then a every locally uniform morphism $X \to Y$ is harmonic.

A motivation?

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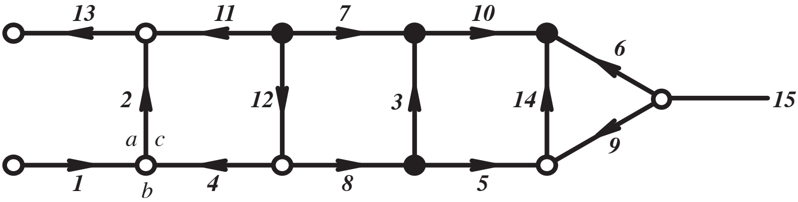
1. Which results of the theory of graph coverings generalize harmonic morphism?
2. Building a bridge between GRAPHS with harmonic morphisms, and COMPLEX ANALYSIS (Riemann Surfaces) with branched coverings (meromorphic mappings),

Example: Heawood problem case $n = 30$, voltages are in $(Z_{30}, +)$,



There are branch-points of index 3 at the vertices of degree 1 and a branch-point of index 2 at the free end of the unique semiedge.

Basic properties

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4. composition of two harmonic morphisms $f : X \to Y$, $g : Y \to Z$ is a harmonic morphism $X \to Z$,
5. If Y has a circulation $\nu : D(Y) \to A$, then it lifts to a circulation $\tilde{\nu} : D(X) \to A$.

Harmonic morphisms with transitive $CT(p)$

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- $X \mapsto X/G$ is regular harmonic morphism iff G is semiregular on darts.
- In the regular case the degree of $x \mapsto [x]$ is $|G|$ as is expected, and the multiplicity $m_v = |G_v|$ at every vertex.

Riemann-Hurwitz equation

Theorem

Let $\psi : X \to Y$ be a harmonic morphism, and let $g = g(X)$, $\gamma = \gamma(Y)$ are the respective topological genera of X and Y. Then

$$g - 1 = deg(\psi)(\gamma - 1) + \sum_{v \in V}(m_v - 1) + \sum_{e \in E}(m_e - 1),$$

where V, E is the set of vertices and the set of edges of X, respectively.

m_e-multiplicity of an edge, $m_e = 1$ for loops and ordinary edges, $m_e = 2$ for semiedges, $g(X)$, $\gamma(Y)$ are the Betti numbers.

Riemann-Hurwitz, regular case

Theorem

(Riemann-Hurwitz, regular case) *Let* $\psi : X \to Y$ *be a regular harmonic morphism,* $G \leq CT(\psi)$*.*
Then

$$g - 1 = |G|\Big((\gamma - 1) + \sum_{v \in V(Y)} (1 - \frac{1}{b_v}) + \sum_{e \in E(Y)} (1 - \frac{1}{b_e})\Big),$$

where b_v*, and* b_e*, for* $v \in V(Y)$*,* $e \in E(Y)$ *is the branch index at the vertex* v*, and the edge* e *of* Y*, respectively.*

Riemann-Hurwitz bound

Theorem

Let G be a semiregular group of automorphisms acting on a connected graph of topological genus $g > 1$. Then $|G| \leq 6(g-1)$. Moreover, the equality holds, if and only if the signature of the action of G is $(0; 2, 3; \emptyset)$, or $(0; 3; 2)$.

There are infinitely many examples coming from arc-regular cubic graphs, for the list up to 300 vertices look the web-page of Marston Conder.

Maximal size of a semiregular group

Theorem

(Scott Corri 2013) *For $g \geq 2$ the maximal size $N(g)$ of a semiregular action of a group on a graph with the topological genus g is bounded by*

$$4(g-1) \leq N(g) \leq 6(g-1).$$

Both bounds are attained for infinitely many values of g.

Wiman's theorem for graphs

Theorem

Let X be a graph of cyclomatic number $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on X. Then $N \leq 2g+2$. The upper bound $N = 2g+2$ is attained for any even g. In this case, the signature of orbifold $X/\mathbb{Z}_N$ is $(0; 2, g+1)$, that is, $X/\mathbb{Z}_N$ is a tree with two branch points of order 2 and $g+1$, respectively.

Harmonic functions on graphs

Denote by $\mathbb{L}_X$ the Laplacian of a graph X.
Let A be an **abelian** group and X be a graph. A function $f : V(X) \to A$ is called **harmonic** if and only if $\mathbb{L}_X(f) = \vec{0}$, that is

$$deg(v) \cdot f(v) - \sum_{x \in D_v} f(Ix^{-1}) = 0,$$

for every vertex v of X.

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In analogy with complex analysis this equation can be understood as follows: The value of f at v is the mean of the discrete integral of the values f of the neighbours of v.

A-Flows and harmonic funcions

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An epimorphism $p : X \to Y$ is harmonic if and only if each harmonic function lifts.

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Principal divisors $= L_X(Div(X)) < Div_0(X)$

Jacobian $Jac(X) = Div_0(X)/Prin(X)$.

Definition of the Jacobian through Z-flows

1. the map $S_{v_0}(f) = f - f(0)$,
2. $S_{v_o}(Div(X)) = Div_0(X) \cong Div^0(X) \cong Z$-flows on X satisfying (2KL),
3. $AJ(f) = [f - f(0)]$ is the **Abel-Jacobi map**, it is an epimorphism $Div(X) \to Jac(X)$, it is a harmonic function,
4. Universality lemma: If $f : V \to A$ is harmonic such that A is generated by the images, then A is an epimorphic image of $Jac(X)$.
5. $Jac(X)$ is the largest group onto which one can define a harmonic map!!!

An algorithm to compute $Jac(X)$

1. Pick up a rooted spanning tree T,
2. For each tree dart set a variable x_i,
3. compute the abstract flows on contree variables using (2KL),
4. form system of $|V|$ linear equations in variables $x_1, \ldots, x_{n-1}$,
5. transform the matrix of the system of equations into the Smith form

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7. the Jacobian lifts and projects along a harmonic morphism

Notation

the Jacobian is called as well the Picard group, the sandpile group, the critical group, the dollar group
Dhar 90, Nagnibeda 97, Biggs 99, Cori-Rossin 00, Baker-Norine 07.

Matrix-Tree theorem

Theorem

The number of spanning trees of a connected graph X is equal to the determinant $det\mathbb{L}_X[u]$, where $\mathbb{L}_X[u]$ is the Laplace matrix with the row and collumn corresponding to a vertex u deleted.

Idea of a proof: Denote $\tau(X)$ the number of spanning trees. Then we have:

$$\tau(X) = \tau(X \setminus e) + \tau(X/e)$$

,

Using standard operations with determinants one can prove the same identity for $det\mathbb{L}_X[u]$:

$$det\mathbb{L}_X[u] = det\mathbb{L}_{X\setminus e}[u] + det\mathbb{L}_{X/e}[u].$$

Harmonic morphisms and Jacobians

Since the $Jac(X)$ is defined in terms of flows, it lifts along any harmonic morphism: $Y \to X$.
It follows that $Jac(X)$ is an epimorphic image of $Jac(Y)$. In fact we have $Jac(X) \leq Jac(Y)$.

A Project/Problem

Problem

To write a unified theory of graph coverings and harmonic coverings with applications. It will require to employ pieces of the Bass-Serre theory of graphs of groups as well as some other known results and of course to find non-trivial applications.
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THANK YOU!