Permutation representations of finite groups and association schemes (Part 8)

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Open problem (Bannai–Ito, 1984)

Not all commutative association schemes are obtained as the set of 2-orbits of a multiplicity-free action.

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \quad (j = 0, 1, \dots, d).$$

Eigenvalues of A_i were denoted by P_{ij}, the entries of the first eigenmatrix:

$$(A_0, \dots, A_d) = (E_0, \dots, E_d) P.$$

Open Problem

Given a commutative association scheme $\mathcal{R}$, do the entries of P belong to a cyclotomic field?

"Yes" if $\mathcal{R} = \Omega^2/G$ for some (multiplicity-free) permutation group.

The pentagon has irrational eigenvalue

The dihedral group of order 10 acts on the pentagon.

$$A_0 = I,\ A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

A_1 has eigenvalues 2 and $\theta_\pm$, where

$$\theta_\pm = \frac{-1 \pm \sqrt{5}}{2}.$$

$$(A_0, A_1, A_2) = (E_0, E_1, E_2) \begin{bmatrix} 1 & 2 & 2 \\ 1 & \theta_+ & \theta_- \\ 1 & \theta_- & \theta_+ \end{bmatrix}.$$

Automorphism group of a field

$$\mathbb{F} = \mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\} = \mathbb{Q}[x]/(x^2 - 5)$$

has automorphism $\sigma : \sqrt{5} \mapsto -\sqrt{5}$. The adjacency algebra

$\mathbb{F}[\mathcal{R}] = \mathbb{F}[A_0, A_1, A_2] = \mathbb{F}[A_1] \subseteq M_5(\mathbb{F})$ admits an automorphism σ acting entrywise.

σ leaves A_0, A_1, A_2 invariant, but it interchanges E_1 and E_2, since

$$(E_0, E_1, E_2) = \frac{1}{5}(A_0, A_1, A_2) \begin{bmatrix} 1 & 2 & 2 \\ 1 & \theta_+ & \theta_- \\ 1 & \theta_- & \theta_+ \end{bmatrix}.$$

$\operatorname{Aut} \mathbb{F}[\mathcal{R}]$ permutes primitive idempotents of $\mathbb{F}[\mathcal{R}]$.

What does $\sigma : \sqrt{5} \mapsto -\sqrt{5}$ permute?

$$(E_0, E_1, E_2) = \frac{1}{5}(A_0, A_1, A_2) \begin{bmatrix} 1 & 2 & 2 \\ 1 & \theta_+ & \theta_- \\ 1 & \theta_- & \theta_+ \end{bmatrix} = \frac{1}{5}(A_0, A_1, A_2) Q,$$

$$\implies \sigma(Q) = QS \quad (S \text{ is a permutation matrix}).$$

Similarly (?)

$$(A_0, A_1, A_2) = (E_0, E_1, E_2) \begin{bmatrix} 1 & 2 & 2 \\ 1 & \theta_+ & \theta_- \\ 1 & \theta_- & \theta_+ \end{bmatrix} = (E_0, E_1, E_2) P,$$

$$\implies \sigma(P) = PS.$$

For the first case, $\sigma \in \operatorname{Aut} \mathbb{F}[\mathcal{R}]$, so σ induces a permutation on the set $\{E_0, E_1, E_2\}$.

Need a proper reasoning for the second case.

Automorphism of the adjacency algebra

In a commutative association scheme $\mathcal{R}$,

$$(A_0, \ldots, A_d) = (E_0, \ldots, E_d)P,$$
$$(E_0, \ldots, E_d) = \frac{1}{|\Omega|}(A_0, \ldots, A_d)Q.$$

Let

$$\mathbb{F} = \mathbb{Q}(\{P_{ij} \mid 0 \leq i, j \leq d\}) = \mathbb{Q}(\{Q_{ij} \mid 0 \leq i, j \leq d\}) \subseteq \mathbb{C}.$$

$$\begin{aligned}
\sigma \in \operatorname{Aut}\mathbb{F} &\implies \sigma \in \operatorname{Aut}(\mathbb{F}[\mathcal{R}], \cdot) \implies \sigma \text{ permutes } \{E_i \mid 0 \leq i \leq d\} \\
&\implies \sigma(Q) = QS_1 \text{ for some permutation matrix } S_1. \\
? &\implies \sigma \in \operatorname{Aut}(\mathbb{F}[\mathcal{R}], \circ) \implies \sigma \text{ permutes } \{A_i \mid 0 \leq i \leq d\} \\
&\implies \sigma(P) = PS_2 \text{ for some permutation matrix } S_2.
\end{aligned}$$

Automorphism of the adjacency algebra

$$\sigma \in \operatorname{Aut}\mathbb{F} \Rightarrow \sigma \in \operatorname{Aut}\mathbb{F}[\mathcal{R}] \Rightarrow \sigma(E_i) = E_{i^x} \ (\exists x \in \operatorname{Sym}(\{0,\dots,d\})).$$

Define

$$\begin{array}{rcl} \sigma' : \mathbb{F}[\mathcal{R}] & \to & \mathbb{F}[\mathcal{R}] \\ \sum_i c_i E_i & \mapsto & \sum_i c_i \sigma(E_i) \end{array} \qquad (\text{Warning: } \sigma' \neq \sigma).$$

If $q_{ij}^k \in \mathbb{Q}$, $E_i \circ E_j = \frac{1}{|\Omega|}\sum_k q_{ij}^k E_k$,

$$\begin{aligned} \sigma'(E_i \circ E_j) &= \frac{1}{|\Omega|}\sum_k q_{ij}^k \sigma(E_k) = \sigma\left(\frac{1}{|\Omega|}\sum_k q_{ij}^k E_k\right) \\ &= \sigma(E_i \circ E_j) = \sigma(E_i) \circ \sigma(E_j) \\ &= \sigma'(E_i) \circ \sigma'(E_j) \qquad \Longrightarrow \ \sigma' \in \operatorname{Aut}(\mathbb{F}[\mathcal{R}], \circ) \\ &\Longrightarrow \ \sigma'(A_i) = A_{i^y} \ (\exists y \in \operatorname{Sym}(\{0,\dots,d\})). \end{aligned}$$

$\sigma \in \mathrm{Aut}(\mathbb{F}[\mathcal{R}], \cdot)$ and $\sigma' \in \mathrm{Aut}(\mathbb{F}[\mathcal{R}], \circ)$

$$\sigma(E_i) = E_{i^x} \implies \sigma(E_0, \ldots, E_d) = (E_0, \ldots, E_d) P_x^\top,$$
$$\sigma'(A_i) = A_{i^y} \implies \sigma'(A_0, \ldots, A_d) = (A_0, \ldots, A_d) P_y^\top.$$

$$|\Omega|(E_0, \ldots, E_d) = (A_0, \ldots, A_d) Q$$
$$\implies |\Omega|(E_0, \ldots, E_d) P_x^\top = (A_0, \ldots, A_d) \sigma(Q)$$
$$\implies Q = \sigma(Q) P_x \implies P = P_x^\top \sigma(P) \implies P_x P = \sigma(P),$$

$$(A_0, \ldots, A_d) = (E_0, \ldots, E_d) P$$
$$\implies \sigma'(A_0, \ldots, A_d) = (\sigma(E_0, \ldots, E_d)) P$$
$$\implies (A_0, \ldots, A_d) P_y^\top = (E_0, \ldots, E_d) P_x^\top P$$
$$\implies (A_0, \ldots, A_d) = (E_0, \ldots, E_d) P_x^\top P P_y$$
$$\implies P = P_x^\top P P_y \implies P_x P = P P_y \implies \sigma(P) = P P_y.$$

Commutativity from $P_x P = \sigma(P) = PP_y$

$\forall \sigma, \tau \in \operatorname{Aut} \mathbb{F}$, $\exists x, y, z, w \in \operatorname{Sym}(\{0, \ldots, d\})$,

$$\sigma(P) = P_x P = PP_y,$$
$$\tau(P) = P_z P = PP_w.$$

$$\begin{aligned}
\sigma\tau(P) &= \sigma(P_z P) = P_z \sigma(P) \\
&= P_z(PP_y) = (P_z P)P_y \\
&= \tau(P)P_y = \tau(PP_y) \\
&= \tau\sigma(P).
\end{aligned}$$

$\operatorname{Aut} \mathbb{F}$ is abelian.

The splitting field of an association scheme

For a commutative association scheme $\mathcal{R}$ with the first eigenmatrix P, the field

$$\mathbb{F} = \mathbb{Q}(\{P_{ij} \mid 0 \leq i, j \leq d\})$$

is called the splitting field, and $\operatorname{Aut} \mathbb{F}$ is called the Galois group, also denoted by $\operatorname{Gal}(\mathbb{F}/\mathbb{Q})$.

Theorem (M. (1991))

If a commutative association scheme $\mathcal{R}$ has rational Krein parameters, then the entries of the eigenmatrix belong to a cyclotomic field.

Proof.

Since $\operatorname{Gal}(\mathbb{F}/\mathbb{Q})$ is abelian, the result follows from the Kronecker–Weber theorem (see the next slide). $\square$

The Kronecker–Weber theorem

Hilbert proved in 1896:

Theorem (Kronecker–Weber)

Let $\mathbb{F}$ be a finite extension of $\mathbb{Q}$ with $\mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ abelian. Then there exists a positive integer ℓ such that $\mathbb{F}$ is isomorphic to a subfield of $\mathbb{Q}(\zeta)$, where $\zeta = \exp \frac{2\pi\sqrt{-1}}{\ell}$.

Is it possible to show that, for a commutative association scheme with splitting field $\mathbb{F}$, $\mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ is abelian without assuming $q_{ij}^k \in \mathbb{Q}$?

The same theorem in a different context

Theorem (M. (1991))

Let

$$\begin{aligned} \mathcal{R} &= \text{ a commutative association scheme,} \\ \mathbb{F} &= \text{ the splitting field of } \mathcal{R}, \\ \mathbb{K} &= \mathbb{Q}(q_{ij}^k) \subseteq \mathbb{F}. \end{aligned}$$

Then

$$\mathrm{Gal}(\mathbb{F}/\mathbb{K}) \subseteq Z(\mathrm{Gal}(\mathbb{F}/\mathbb{Q})).$$

In particular, if $\mathbb{K} = \mathbb{Q}$, then $\mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ is abelian, and hence $\mathbb{F}$ is contained in a cyclotomic field.

Essentially the same statement was proved independently:
A. Coste and T. Gannon, Remarks on Galois symmetry in rational conformal field theories, Phys. Lett. B 323 (1994) 316–321.

Summary

- If $\mathcal{R} = \Omega^2/G$ is commutative, then the eigenvalues of adjacency matrices belong to a cyclotomic field.
- It is unknown whether the same statement holds for commutative association schemes in general.
- If $q_{ij}^k \in \mathbb{Q}$, then this holds true.
- The Coxeter graph (anti flags of the Fano plane) is an association scheme with 28 points, rank 5.

$$\mathbb{F} = \mathbb{K} = \mathbb{Q}(\sqrt{2}).$$

References

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