Permutation representations of finite groups and association schemes (Part 3)

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Contents

Review

In the previous lecture, we discussed the adjacency algebra $\mathbb{C}[\mathcal{R}]$ of an association scheme $\mathcal{R}$. In this lecture we always take $\mathbb{F} = \mathbb{C}$.

If $\mathcal{R}$ is the set of 2-orbits of an action $(G; \Omega)$, then

$$\mathbb{C}[\mathcal{R}] = V_{\mathbb{C}}(G, \Omega)$$

which is the centralizer in $M_\Omega(\mathbb{C})$ of

$$\{P_g \mid g \in G\}.$$

That is, the two algebras

$$\mathbb{C}[\mathcal{R}] \text{ and } \mathbb{C}[\{P_g \mid g \in G\}]$$

centralize each other.

Note that $\mathbb{C}[\{P_g \mid g \in G\}]$ is not a group algebra since P_g $(g \in G)$ are not necessarily linearly independent.

Then what is $\dim \mathbb{C}[\{P_g \mid g \in G\}]$?

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Then what is $\dim \mathbb{C}[\{P_g \mid g \in G\}]$?

$*$-algebras

For $A \in M_n(\mathbb{C})$, write $A^* = \overline{A}^\top$.

Definition

A subalgebra $\mathcal{A}$ of $M_n(\mathbb{C})$ is a $*$-algebra if

$$A \in \mathcal{A} \implies A^* \in \mathcal{A}.$$

If $(G;\Omega)$ is an action, then $\{P_g \mid g \in G\}$ is closed under transposition: $P_g^\top = P_{g^{-1}}$. Indeed,

$$((P_g)^\top)_{\alpha\beta} = (P_g)_{\beta\alpha} = \delta_{\beta^g\alpha} = \delta_{\alpha^{g^{-1}}\beta} = (P_{g^{-1}})_{\alpha\beta}.$$

Thus $\mathbb{C}[\{P_g \mid g \in G\}]$ is a $*$-algebra.

If $\mathcal{R}$ is an association scheme, then $\{A(R) \mid R \in \mathcal{R}\}$ is closed under transposition. Thus $\mathbb{C}[\mathcal{R}]$ is a $*$-algebra.

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The standard module

Let $\mathcal{A} \subseteq M_n(\mathbb{C})$ be a $*$-algebra. Since $\mathbb{C}^n$ is a left $M_n(\mathbb{C})$-module, it is also an $\mathcal{A}$-module, which is called the standard module.

Recall that an $\mathcal{A}$-module W is irreducible if it has no nontrivial $\mathcal{A}$-submodule.

Proposition

Let $\mathbb{C}^n \supseteq V$ be an $\mathcal{A}$-submodule of a $*$-algebra $\mathcal{A} \subseteq M_n(\mathbb{C})$. Then $V^\perp$ is also an $\mathcal{A}$-submodule.

Thus, the standard module $\mathbb{C}^n$ is an orthogonal direct sum of irreducible $\mathcal{A}$-modules. This means that, $\exists$ unitary matrix U,

$$U^* \mathcal{A} U = \text{algebra of block diagonal matrices}$$

and each block (say, size $m \times m$) acts on $\mathbb{C}^m$ irreducibly, hence isomorphic to $M_m(\mathbb{C})$ by Wedderburn's theorem.

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Wedderburn's theorem (cf. Rotman's book)

Upon block-diagonalizing $\mathcal{A}$, we find a block $\mathcal{B} \subseteq M_m(\mathbb{C})$ such that

$\mathbb{C}^m$ is an irreducible $\mathcal{B}$-module, and it is faithful

$$\implies \mathcal{B} \text{ is simple } (\nexists \text{ nontrivial ideal})$$

$$\implies \mathcal{B} = \bigoplus_{i=1}^{k} L_i \quad \text{pairwise isomorphic minimal left ideals}$$

$$\begin{aligned}
\mathcal{B} &\cong \mathrm{End}_{\mathcal{B}}(\mathcal{B})^{op} \\
&\cong \mathrm{End}_{\mathcal{B}}\Big(\bigoplus_{i=1}^{k} L_i\Big)^{op} \\
&\cong M_k(\mathbb{C})^{op} \quad \text{(by Schur's lemma)} \\
&\cong M_k(\mathbb{C})
\end{aligned}$$

$$\implies \text{the only irreducible } \mathcal{B}\text{-module has } \dim = k$$

$$\implies m = k, \text{ so } \mathcal{B} = M_m(\mathbb{C}).$$

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$\implies$ $m = k$, so $\mathcal{B} = M_m(\mathbb{C})$.

Block diagonalization

$$U^*\mathcal{A}U = \begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & M_m(\mathbb{C}) & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix} \quad \begin{bmatrix} \vdots \\ \bigoplus \\ \mathbb{C}^m \\ \bigoplus \\ \vdots \end{bmatrix} = \mathbb{C}^n.$$

Be careful! Isomorphic modules may appear more than once. What if

$$U^*\mathcal{A}U = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \mid A \in M_m(\mathbb{C}) \right\}? \quad \neq \begin{bmatrix} M_m(\mathbb{C}) & 0 \\ 0 & M_m(\mathbb{C}) \end{bmatrix}.$$

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Representation of an algebra

A (matrix) representation of a $\mathbb{C}$-algebra $\mathcal{A}$ is a $\mathbb{C}$-algebra homomorphism $\Delta : \mathcal{A} \to M_m(\mathbb{C})$. This is equivalent to giving an $\mathcal{A}$-module of dimension m.

Since we only consider $*$-algebras, we require a representation to be $*$-representation, that is

$$\Delta(A^*) = \Delta(A)^*.$$

Block diagonalization by a unitary matrix satisfies this requirement, since

$$(U^*AU)^* = U^*A^*U.$$

Our representations are of the form "first conjugating by a unitary matrix, then extracting a block".

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Block diaognalization

The block diaognalization of a $*$-algebra $\mathcal{A} \subseteq M_n(\mathbb{C})$ is described as

$$U^* \mathcal{A} U = \begin{bmatrix} \ddots & & \\ & \boxed{\begin{matrix} \Delta_i(A) & & \\ & \ddots & \\ & & \Delta_i(A) \end{matrix}} & \\ & & \ddots \end{bmatrix}$$

($\Delta_i(A)$ is repeated m_i times).
$\Delta_i : \mathcal{A} \to M_{n_i}(\mathbb{C})$ is a $*$-representation, and Δ_i is surjective by Wedderburn's theorem.

$$\left\{ \boxed{\begin{matrix} \Delta_i(A) & & \\ & \ddots & \\ & & \Delta_i(A) \end{matrix}} \;\middle|\; A \in \mathcal{A} \right\} \cong M_{n_i}(\mathbb{C}).$$

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Centralizer (= matrices which commute with given ones)

What is the centralizer of

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Centralizer (I)

Centralizer of $M_n(\mathbb{C})$ is $\mathbb{C}I$ (scalar matrices).

The centralizer of

$$U^*\mathcal{A}U = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \mid A \in M_d(\mathbb{C}) \right\}?$$

is

$$\begin{bmatrix} \mathbb{C}I & \mathbb{C}I \\ \mathbb{C}I & \mathbb{C}I \end{bmatrix} = \left\{ \begin{bmatrix} a_{11}I & a_{12}I \\ a_{21}I & a_{22}I \end{bmatrix} \mid \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\mathbb{C}) \right\}.$$

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in $M_{dm}(\mathbb{C})$ is

$$\left\{ \begin{bmatrix} a_{11}I_d & \cdots & a_{1m}I_d \\ \vdots & \ddots & \vdots \\ a_{m1}I_d & \cdots & a_{mm}I_d \end{bmatrix} \mid [a_{ij}] \in M_m(\mathbb{C}) \right\}.$$

$M_d(\mathbb{C}) \otimes I_m$ and $I_d \otimes M_m(\mathbb{C})$ centralize each other.

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where $\Delta_i(A)$ is repeated m_i times, $A \in M_{n_i}(\mathbb{C})$, is

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Centralizer(Centralizer($\mathcal{A}$)) $= \mathcal{A}$ (The double centralizer theorem).

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The Johnson scheme $J(n,2)$

It is the action $(S_n;\Omega)$, where

$$\Omega = \{\alpha \mid \alpha \subseteq \{1,\dots,n\},\ |\alpha| = 2\}.$$

Its adjacency algebra $\mathbb{C}[\mathcal{R}] \subseteq M_\Omega(\mathbb{C})$ is isomorphic to $\mathbb{C}^3$. The (block) diagonal form is

$$\begin{bmatrix} \ddots & & \\ & M_{n_i}(\mathbb{C}) \otimes I_{m_i} & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \mathbb{C}I_{m_0} & & \\ & \mathbb{C}I_{m_1} & \\ & & \mathbb{C}I_{m_2} \end{bmatrix}$$

$$\mathbb{C}[\{P_g \mid g \in S_n\}] \cong \begin{bmatrix} M_{m_0}(\mathbb{C}) & & \\ & M_{m_1}(\mathbb{C}) & \\ & & M_{m_2}(\mathbb{C}) \end{bmatrix}$$

multiplicity-free $\iff$ $n_i = 1$ $(\forall i)$.

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The group algebra $\mathbb{C}[G]$ (I)

The group algebra of a group G is the algebra with basis indexed by the elements of G, such that multiplication is done just as in G. The algebra of the right regular representation $(G_R; G)$

$$\mathcal{A} = \mathbb{C}[\{P_g \mid g \in G\}]$$

is such an algebra. Since this is a $*$-algebra, its block diagonal form is

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The group algebra $\mathbb{C}[G]$ satisfies $m_i = n_i$

$$U^* \mathcal{A} U = \begin{bmatrix} \ddots & & \\ & \boxed{M_{n_i}(\mathbb{C}) \otimes I_{m_i}} & \\ & & \ddots \end{bmatrix}$$

The size $|G|$ of the matrix is $\sum n_i m_i$.
The dimension of the algebra is $|G| = \sum n_i^2$.
The centralizer of $\mathcal{A}$ is the algebra of left regular representation, its dimension is

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Equality in Cauchy–Schwarz inequality forces $m_i = n_i$ $(\forall i)$. This proves

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The centralizer of $\mathcal{A}$ is the algebra of left regular representation, its dimension is

$$|G| = \sum m_i^2.$$

Equality in Cauchy–Schwarz inequality forces $m_i = n_i$ $(\forall i)$. This proves

$$\mathbb{C}[G] \cong \begin{bmatrix} \ddots & & \\ & \boxed{M_{n_i}(\mathbb{C}) \otimes I_{n_i}} & \\ & & \ddots \end{bmatrix}.$$

Summary

The centralizer algebra $V_{\mathbb{C}}(G, \Omega)$, or more generally, the adjacency algebra $\mathbb{C}[\mathcal{R}]$ of an association scheme, is (abstractly) isomorphic to

$$\bigoplus_i M_{n_i}(\mathbb{C}).$$

In terms of block diagonal matrices, $M_{n_i}(\mathbb{C})$ appear m_i times in the diagonal.

In particular, the group algebra:

$$\begin{aligned}\mathbb{C}[G] &\cong \text{the algebra of the right regular representation}\\ &\cong \text{the algebra of the left regular representation}\\ &= \text{centralizer of the right regular representation}\end{aligned}$$

is also isomorphic to the direct sum of the full matrix algebras.

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Corollary 8.65 (Molien). *If G is a finite group and k is an algebraically closed field whose characteristic does not divide $|G|$, then*

$$kG \cong \mathrm{Mat}_{n_1}(k) \times \cdots \times \mathrm{Mat}_{n_m}(k).$$

Proof. By Maschke's theorem, kG is a semisimple ring, and its simple components are isomorphic to matrix rings of the form $\mathrm{Mat}_n(\Delta)$, where Δ arises as $\mathrm{End}_{kG}(L)^{\mathrm{op}}$ for some minimal left ideal L in kG. Therefore, it suffices to show that $\mathrm{End}_{kG}(L)^{\mathrm{op}} = \Delta = k$.

Now $\mathrm{End}_{kG}(L)^{\mathrm{op}} \subseteq \mathrm{End}_k(L)^{\mathrm{op}}$, which is finite-dimensional over k because L is; hence, $\Delta = \mathrm{End}_{kG}(L)^{\mathrm{op}}$ is finite-dimensional over k. Each $f \in \mathrm{End}_{kG}(L)$ is a kG-map, hence is a k-map; that is, $f(au) = af(u)$ for all $a \in k$ and $u \in L$. Therefore, the map $\varphi_a \colon L \to L$, given by $u \mapsto au$, commutes with f; that is, k (identified with all φ_a) is contained in $Z(\Delta)$, the center of Δ. If $\delta \in \Delta$, then δ commutes with every element in k, and so $k(\delta)$, the subdivision ring generated by k and δ, is a (commutative) field. As Δ is finite-dimensional over k, so is $k(\delta)$; that is, $k(\delta)$ is a finite extension of the field k, and so δ is algebraic over k, by Proposition 3.117. But k is algebraically closed, so that $\delta \in k$ and $\Delta = k$. •