Permutation representations of finite groups and association schemes (Part 4)

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G2R2, Novosibirsk

Review

In the previous lecture, we discussed the structure of the adjacency algebra $\mathcal{A} = \mathbb{C}[\mathcal{R}]$ of an association scheme $\mathcal{R}$, or more generally, $*$-algebras in $M_\Omega(\mathbb{C})$. We consider the special case where $\mathcal{A}$ is commutative.

The (block) diagonal form is, when $n_i = 1$ $(\forall i)$,

$$\begin{bmatrix} \ddots & & \\ & M_{n_i}(\mathbb{C}) \otimes I_{m_i} & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \mathbb{C} I_{m_i} & \\ & & \ddots \end{bmatrix}$$

Consequence of an elementary fact that pairwise commutative normal matrices are simultaneously diagonalizable by a unitary matrix.

$$\mathcal{A} = \cong \mathbb{C}^r \quad \text{for some } r.$$

$\exists$ uniquely determined basis $E_1, \ldots, E_r$ with $E_i E_j = \delta_{ij} E_i$.

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Since $U^*\mathcal{A}U$ consists of diagonal matrices...

$$\mathcal{A} = U \begin{bmatrix} \ddots & & \\ & \mathbb{C} I_{m_i} & \\ & & \ddots \end{bmatrix} U^*$$

has a basis of primitive idempotents

$$E_i = U \begin{bmatrix} 0 & & \\ & I_{m_i} & \\ & & 0 \end{bmatrix} U^* \quad (i = 1, \ldots, r),$$

satisfying $E_i E_j = \delta_{ij} E_i$. Moreover, $E_i = E_i^*$.

Every idempotent $E \in \mathcal{A}$ is of the form

$$E = \sum_{i \in S} E_i \quad (\text{for some } S \subseteq \{1, \ldots, r\}),$$

$$\operatorname{rank} E = \sum_{i \in S} \operatorname{rank} E_i.$$

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Primitive idempotents

Since

$$\sum_{R \in \mathcal{R}} A(R) = J \quad \text{all-one matrix}$$

and

$$\frac{1}{|\Omega|} J \quad \text{is an idempotent of rank 1,}$$

we may assume

$$E_0 = \frac{1}{|\Omega|} J, E_1, \ldots, E_d \quad (\dim \mathcal{A} = d + 1)$$

is a basis of $\mathcal{A}$.

Adjacency matrices

Since $\mathcal{R}$ is an association scheme on the set Ω, the diagonal relation $1_\Omega \in \mathcal{R}$.

$$\{A(R) \mid R \in \mathcal{R}\} = \{A_0 = I, A_1, \dots, A_d\}.$$

There exists a nonsingular $(d+1) \times (d+1)$ matrix P (called the first eigenmatrix) such that

$$(A_0, \dots, A_d) = (E_0, \dots, E_d)P$$

$$A_j = \sum_i P_{ij} E_i.$$

Since $E_i E_j = \delta_{ij} E_i$.

$$A_j A_{j'} = \sum_i P_{ij} P_{ij'} E_i.$$

multiplication in $\mathcal{A}$ = entrywise product of column vectors of P.

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Entrywise product

Recall the Schur–Hadamard product

$$(X \circ Y)_{ij} = X_{ij} Y_{ij}.$$

$$A_i \circ A_j = \delta_{ij} A_i.$$

Duality

$$\begin{array}{ccc} \{A_i\} & & \{E_i\} \\ \circ & \text{exchange} & \cdot \\ P & & P^{-1} \end{array}$$

Next we discuss the missing piece

$$\begin{array}{ll} A_i \circ A_j = \delta_{ij} A_i & E_i E_j = \delta_{ij} E_i \\ A_i A_j = \sum_k p_{ij}^k A_k & E_i \circ E_j = ? \end{array}$$

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The second eigenmatrix Q

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\begin{aligned}
(E_0, \dots, E_d) &= (A_0, \dots, A_d) P^{-1} \\
&= \frac{1}{|\Omega|} (A_0, \dots, A_d) Q.
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$$E_j = \frac{1}{|\Omega|} \sum_i Q_{ij} A_i.$$

Since $A_i \circ A_j = \delta_{ij} A_i$,

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$\circ$-multiplication in $\mathbb{C}[\mathcal{R}]$ $=$ entrywise product of column vectors of Q up to scalar.

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Krein parameters

$$E_i \circ E_j = \frac{1}{|\Omega|} \sum_k q_{ij}^k E_k.$$

The scalars q_{ij}^k are called Krein parameters.

$$\begin{aligned} E_i^2 = E_i \ (\forall i) &\implies \text{eigenvalues are } 0, 1 \\ &\implies x^* E_i x \geq 0 \ (\forall i,\ \forall x \in \mathbb{C}^\Omega) \\ &\overset{\text{def}}{\iff} E_i \succeq 0. \end{aligned}$$

$$\begin{aligned} E_i, E_j \succeq 0 &\implies E_i \otimes E_j \succeq 0 \\ &\implies E_i \circ E_j \succeq 0 \\ &\implies \sum_k q_{ij}^k E_k \succeq 0 \quad \text{(indeed)} \qquad \sum_k c_k E_k \succeq 0 \\ &\implies q_{ij}^k \geq 0 \qquad\qquad\qquad\qquad \iff c_k \geq 0 \ (\forall k). \end{aligned}$$

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Spherical representation: the hypercube

A representation of an association scheme $\mathcal{R}$ on Ω is a mapping ϕ from Ω to the unit sphere $S^{m-1} \subseteq \mathbb{R}^m$ which preserve $\mathcal{R}$:

$$\forall R \in \mathcal{R},\ \exists \theta_R,\ \forall(\alpha,\beta) \in R,\ (\phi(\alpha),\phi(\beta)) = \cos\theta_R.$$

The hypercube

$$\frac{1}{\sqrt{m}}\{1,-1\}^m \subseteq S^{m-1} \subseteq \mathbb{R}^m$$

forms an association scheme whose relations are defined by the Euclidean (or angular, or Hamming) distance. This definition itself gives a spherical representation.

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Equivalently, the Gram matrix

$$F = [(\phi(\alpha), \phi(\beta))]_{\alpha\beta} \in M_\Omega(\mathbb{R}) \cap \mathbb{C}[\mathcal{R}]$$

is symmetric positive semidefinite, and diagonals are 1.

What are the positive semidefinite real symmetric matrices in $\mathbb{C}[\mathcal{R}]$?

The primitive idempotents are Hermitian positive semidefinite.

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They are real symmetric matrices of the form

$$\sum_i c_i E_i \quad \text{with } c_i \geq 0 \quad (\forall i).$$

Since

$$\{A_i^\top \mid i = 0, \dots, d\} = \{A_i \mid i = 0, \dots, d\},$$
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there exist permutations $i \mapsto i'$, $i \mapsto \hat{\imath}$ such that

$$A_i^\top = A_{i'}, \quad E_i^\top = E_{\hat{\imath}}.$$

So it is sensible to consider the symmetric part of $\mathbb{C}[\mathcal{R}]$:

$$\tilde{R} = \{A(R \cup R^\top) \mid R \in \mathcal{R}\}$$

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Symmetrization

Given a commutative association scheme $\mathcal{R}$,

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forms an association scheme, called the symmetrization of $\mathcal{R}$.

This is because, if $\{E_i\}$ are the primitive idempotents of $\mathbb{C}[\mathcal{R}]$, then $\mathbb{C}[\tilde{R}]$ has idempotent basis

$$\{E_i \mid E_i = E_i^\top\} \cup \{E_i + E_i^\top \mid E_i \neq E_i^\top\}.$$

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A spherical representation of $J(n, 2)$

$$\begin{aligned}\Omega &= \{\{i,j\} \mid i,j \in \{1,\dots,n\},\ i<j\} \\ &= \{\alpha \mid \alpha \subseteq \{1,\dots,n\},\ |\alpha| = 2\}.\end{aligned}$$

$$\begin{aligned}R_0 &= \{(\alpha,\alpha) \mid \alpha \in \Omega\}, \\ R_1 &= \{(\alpha,\beta) \mid \alpha,\beta \in \Omega,\ |\alpha \cap \beta| = 1\}, \\ R_2 &= \{(\alpha,\beta) \mid \alpha,\beta \in \Omega,\ \alpha \cap \beta = \emptyset\}.\end{aligned}$$

$$\begin{array}{rcl}\phi : \Omega & \to & S^{n-1} \subseteq \mathbb{R}^n \\ \{i,j\} & \mapsto & \frac{1}{\sqrt{2}}(e_i + e_j)\end{array}$$

$$\arccos(\phi(\alpha), \phi(\beta)) = 0, \frac{\pi}{3}, \frac{\pi}{2}.$$

This is realized by the Gram matrix

$$(n-1)E_0 + (\frac{n}{2} - 1)E_1.$$

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or equivalently, $P_{0i} = k_i$,

$m_j \stackrel{\text{def}}{=} \operatorname{rank} E_j$.

$$\operatorname{tr}(A_i \overline{E_j}) = \operatorname{tr}(A_i E_j^\top) = \sum_{\alpha,\beta} (A_i \circ E_j)_{\alpha\beta} = \frac{|\Omega| k_i}{|\Omega|} Q_{ij},$$

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$$\mathrm{tr}(A_i \overline{E_j}) = \mathrm{tr}(A_i E_j^\top) = \sum_{\alpha,\beta} (A_i \circ E_j)_{\alpha\beta} = \frac{|\Omega| k_i}{|\Omega|} Q_{ij},$$

$$\mathrm{tr}(A_i \overline{E_j}) = \overline{\mathrm{tr}(A_i E_j)} = \overline{P_{ji}\,\mathrm{tr}(E_j)} = \overline{P_{ji}} m_j.$$

$PQ = |\Omega| I,\ Q_{ij} = \overline{P_{ji}}\, \mathrm{rank}\, E_j / k_i$

$$(A_0, \dots, A_d) = (E_0, \dots, E_d) P,$$
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$A_i J = k_i J$ (row sum k_i), equivalently $A_i E_0 = k_i E_0$,
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Delsarte's bound

Suppose $R_1 \in \mathcal{R}$ (commutative association scheme), $R_1^\top = R_1 \neq 1_\Omega$, the graph (Ω, R_1) has a clique of size c.

The characteristic vector u of that clique satisfies

$$u^\top u = c, \quad u^\top A_1 u = c(c-1), \quad u^\top A_i u = 0 \quad (i > 1).$$

Since $Q_{ij} = \overline{P_{ji}} m_j / k_i$, and $P_{10} = 1$, $P_{11} = \overline{P_{11}}$,

$$\begin{aligned} 0 \leq u^\top |\Omega| E_1 u &= u^\top \sum_i Q_{ij} A_i u \\ &= Q_{01} u^\top u + Q_{11} u^\top A_1 u = c Q_{01} + c(c-1) Q_{11} \\ &= c m_1 \left(1 + (c-1) \frac{P_{11}}{k_1}\right), \end{aligned}$$

$$c \leq 1 + \frac{k_1}{\theta_{\min}} \quad \text{where } \theta_{\min} = \text{ smallest eigenvalue of } A_1.$$

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Summary

For a commutative association scheme $\mathcal{R}$, the adjacency algebra $\mathbb{C}[\mathcal{R}]$ has two canonical bases $\{A_i\}$ and $\{E_i\}$, and they are related by the first and second eigenmatrices P, Q:

$$(A_0, \dots, A_d) = (E_0, \dots, E_d)P,$$

$$(E_0, \dots, E_d) = \frac{1}{|\Omega|}(A_0, \dots, A_d)Q,$$

$$\frac{Q_{ij}}{\operatorname{rank} E_j} = \frac{\overline{P_{ji}}}{k_i}, \quad A_i J = k_i J.$$

The structure constants for the algebra $\mathbb{C}[\mathcal{R}]$ are

$$A_i A_j = \sum_k p_{ij}^k A_k \quad \text{(intersection numbers)},$$

$$E_i E_j = \frac{1}{|\Omega|} \sum_k q_{ij}^k E_k \quad \text{(Krein parameters)}.$$