On triple intersection numbers of association schemes

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based on joint work in progress with
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Association Schemes

Let X be a finite set of v elements, and $\mathcal{R}$ be a partition of $X \times X$ into $D+1$ symmetric binary relations $R_0, R_1, ..., R_D$.

$(X, \mathcal{R})$ is a symmetric **association scheme** if:

- one of the relations, say R_0, is the identity;
- $\forall i, j, k$: for $(x, y) \in R_k$, the number of elements $z \in X$ s.t.

$$(x, z) \in R_i \quad \text{and} \quad (y, z) \in R_j$$

is a constant denoted by $p_{ij}^k = p_{ji}^k$, which does not depend on the particular choice of $(x, y) \in R_k$.

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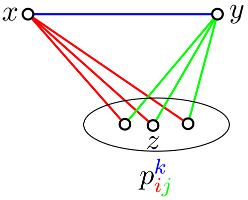

Adjacency matrices

For every relation R_i, define the adjacency matrix $\mathsf{A}_i \in \mathbb{R}^{X \times X}$:

$$(\mathsf{A}_i)_{x,y} := \begin{cases} 1 \text{ if } (x,y) \in R_i, \\ 0 \text{ if } (x,y) \notin R_i, \end{cases}$$

Eigenspaces of $\mathsf{A}_0 = \mathsf{I}, \mathsf{A}_1, \ldots, \mathsf{A}_D$:

$\mathbb{R}^X =$	$\mathbf{W}_0 \oplus$	$\mathbf{W}_1 \oplus$	$\ldots \oplus$	$\mathbf{W}_D$
A_0	P_{00}	P_{01}	$\ldots$	P_{0D}
A_1	P_{10}	P_{11}	$\ldots$	P_{1D}
$\ldots$	$\ldots$	$\ldots$	P_{ij}	$\ldots$
A_D	P_{D0}	P_{D1}	$\ldots$	P_{DD}

Orthogonal projections:

$$\mathsf{E}_j : \mathbb{R}^X \mapsto \mathbf{W}_j$$

$$\mathsf{A}_i = \sum_{j=0}^{D} P_{ij} \mathsf{E}_j$$

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Two bases of the Bose-Mesner algebra

$$\langle \mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_D \rangle = \langle \mathsf{E}_0, \mathsf{E}_1, \ldots, \mathsf{E}_D \rangle$$

$$\mathsf{A}_i = \sum_{j=0}^{D} P_{ij} \mathsf{E}_j, \qquad \mathsf{E}_j = \sum_{i=0}^{D} Q_{ji} \mathsf{A}_i$$

(Q_{ji} — dual eigenvalues)

with respect to the standard matrix multiplication:

$$\mathsf{A}_i \mathsf{A}_j = \sum_{k=0}^{D} p_{ij}^k \mathsf{A}_k, \qquad \mathsf{E}_i \mathsf{E}_j = \delta_{ij} \mathsf{E}_i$$

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P-polynomial association schemes

An association scheme is called P-**polynomial** or **metric** if:

$$\mathsf{A}_i = v_i(\mathsf{A}_1),$$

(with respect to some ordering of $R_0, R_1, \ldots, R_D$)
for some polynomials v_i, $i = 0, 1, \ldots, D$, of degree i.

This is equivalent to the **triangle inequality**:

$$p_{ij}^k = 0 \quad \text{whenever} \quad |i-j| > k \quad \text{or} \quad i+j < k,$$

in which case we can define the distance function ∂ on X:

$$\partial(x, y) = i \Leftrightarrow (x, y) \in R_i.$$

and consider a graph defined on X with

$$x \sim y \Leftrightarrow \partial(x, y) = 1 \Leftrightarrow (x, y) \in R_1.$$

Such a graph is said to be **distance-regular**.

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For a P-polynomial association scheme:

$$\mathsf{A}_i = v_i(\mathsf{A}_1)$$

By dualising the P-polynomiality, we say that an association scheme is Q-**polynomial** or **cometric** if

$$\mathsf{E}_j = \frac{1}{|X|} v_j^*(|X| \cdot \mathsf{E}_1),$$

(with respect to some ordering of E_j's)

where v_j^* is a $\circ$-polynomial of degree j.

Q-polynomial association schemes were introduced by Delsarte in '*An Algebraic Approach to the Association Schemes of Coding Theory*' (1973).

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P vs. $(P\&Q)$ vs. Q

(we consider only primitive schemes here)

P: many examples that are only P-polynomial (including infinite families).

$(P\&Q)$: Many important examples are here: Hamming schemes, Johnson schemes, Grassmann schemes...

Bannai's Conjecture (early 1980s):

(1) If diameter D is large enough, then a P-polynomial association scheme is Q-polynomial, and vice versa.

(2) All (P **and** Q)-polynomial association schemes are known.

Q: very few cometric association schemes are known

- only 7 examples in: W.J. Martin, H. Tanaka, "Commutative association schemes" (Europ. J. Combin., 2009)
- nice examples were recently found by Gavin D. King (2018): $PSp(8,2)/S_{10}$, $O^{-}(10,2)/S_{12}$, $O_8^{+}(3).2/O_8^{+}(2).2$, ...

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(some of known) Q-Polynomial association schemes

- the block scheme of the $4-(11,5,1)$ Witt design, with $D=3$, $|X|=66$;
- the block scheme of the $5-(24,8,1)$ Witt design, with $D=3$, $|X|=729$;
- a spherical design derived from the Leech lattice with $D=3$, $|X|=2025$;
- the block scheme of a $4-(47,11,48)$ design arising from codewords of weight 11 in a certain quadratic residue code of length 47, with $D=3$, $|X|=4324$;
- the "antipodal" quotient of the association scheme on shortest vectors of the Leech lattice, with $D=3$, $|X|=98280$;
- a spherical design derived from the Leech lattice with $D=4$, $|X|=7128$;
- another derived spherical design arising among the shortest vectors of the Leech lattice, with $D=5$, $|X|=47104$.

Table of feasible parameters

- Very few (primitive) cometric association schemes are known.
- If we want to find new schemes, it is useful to have a list of their feasible parameters, i.e., those that:
  - have non-negative integral p_{ij}^k,
  - have non-negative integral eigenvalue multiplicities,
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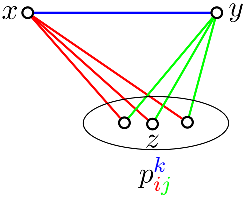
The intersection numbers



Triple intersection numbers

Fix a triple of vertices u, v, w such that

$$(u, v) \in R_W, \quad (u, w) \in R_V, \quad (v, w) \in R_U.$$

Denote a **triple intersection number** $[\ell, m, n]$:

$$[\ell, m, n]_{u,v,w} = |\{z : \ (u, z) \in R_\ell, (v, z) \in R_m, (w, z) \in R_n\}|.$$

Unlike the intersection numbers, the triple intersection numbers in general depend on the choice of u, v, w (not only on U, V, W).

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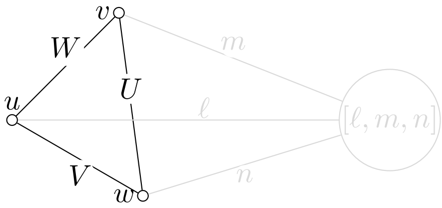

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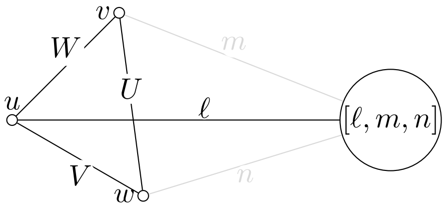

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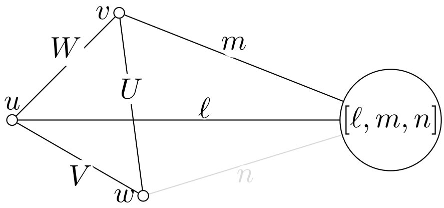

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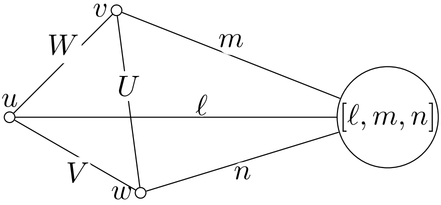

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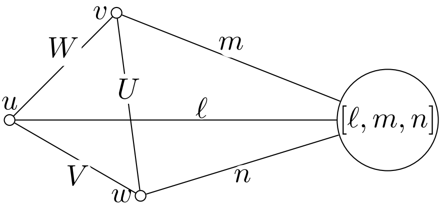

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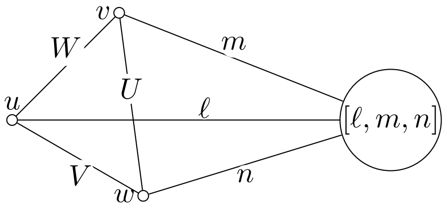

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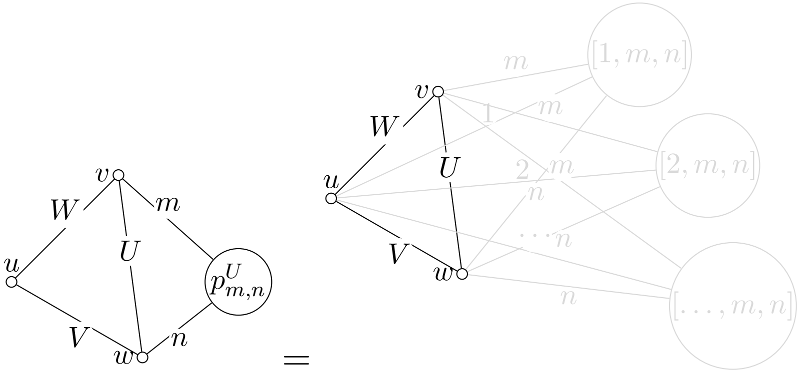

$$\sum_{i=1}^{D}[i,m,n] = p^U_{m,n} - [0,m,n], \quad [0,m,n] = \delta_{m,W}\,\delta_{n,V},$$

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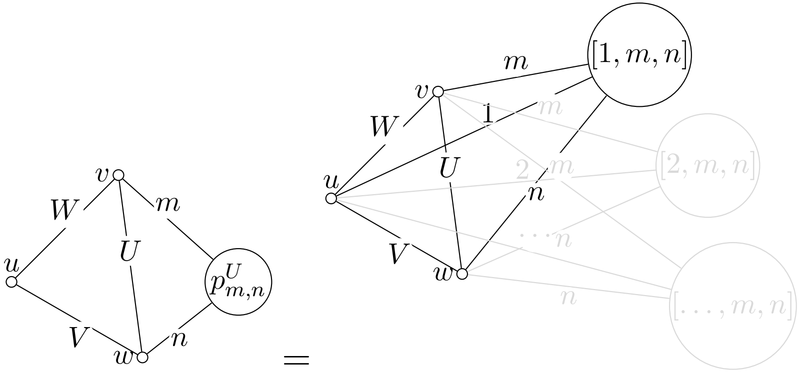

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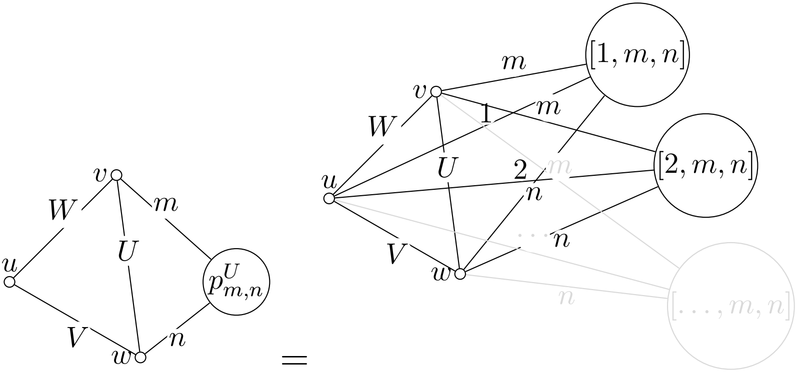

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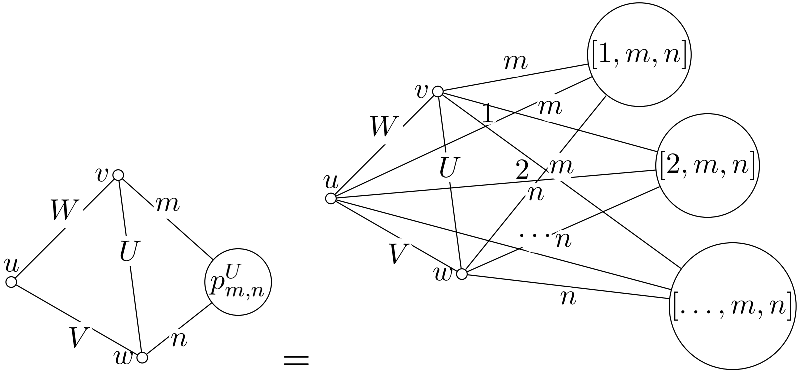

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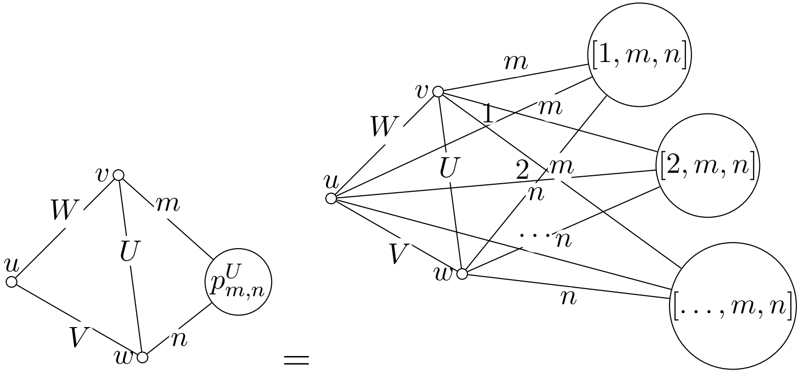

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We obtain $3D^2$ linear Diophantine equations:

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with D^3 unknowns $\in \mathbb{Z}_{\geq 0}$ (with respect to fixed triple u, v, w).

Furthermore, we can use the triangle inequality:

$$p^k_{ij} = 0 \quad \text{whenever} \quad |i-j| > k \quad \text{or} \quad i+j < k$$

to conclude that certain triple intersection numbers must be 0.

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Vanishing of Krein parameters

Recall:

$$\mathsf{A}_i = \sum_{j=0}^{D} P_{ij} \mathsf{E}_j, \qquad \mathsf{E}_j = \sum_{i=0}^{D} Q_{ji} \mathsf{A}_i,$$

Theorem (BCN, Thm 2.3.2)

Let $(X, \mathcal{R})$ be an association scheme with Krein parameters q_{ij}^k $(0 \le i, j, k \le D)$. Then $q_{ij}^k = 0$ implies $q_{ik}^j = 0$ and $q_{kj}^i = 0$ and

$$q_{ij}^k = 0 \Leftrightarrow \sum_{r,s,t=0}^{D} Q_{ir} Q_{js} Q_{kt} [r, s, t]_{u,v,w} = 0$$

for any triple $u, v, w \in X$.

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Smallest open case

The smallest open case from Williford's table is a putative scheme on 91 points whose parameters were found by van Dam ("Three-class association schemes" // J. Alg. Comb., 1999).

Parameters	∃	v	m1	Krein Array	multiplicities	valencies	2nd Q	P	DRG	SRG	Ex	Comm
<27,6>	!	27	6	{6,4,2; 1,2,3}	1,6,12,8	1,6,12,8	-	0123	{6,4,2;1,2,3}	-		Hamming
<35,6>	!	35	6	{6,49/10,7/2; 1,21/10,7/2}	1,6,14,14	1,12,18,4	-	0123,0312	{12,6,2;1,4,9}	<35,18,9,9>	Y	Johnson
<56,7>	!	56	7	{7,256/45,98/25; 1,448/225,14/5}	1,7,20,28	1,15,30,10	-	0123	{15,8,3;1,4,9}	-		Johnson
<64,7>	!	64	7	{7,6,5; 1,2,3}	1,7,21,35	1,21,35,7	0231	0123,0312	{21,10,3;1,6,15}	<64,35,18,20>	Y	Halved 7
<64,9>	2!	64	9	{9,6,3; 1,2,3}	1,9,27,27	1,9,27,27	-	0123	{9,6,3;1,2,3}	<64,27,10,12>		Hamming H(
<64,21>	!	64	21	{21,10,3; 1,6,15}	1,21,35,7	1,7,21,35	0312	0123,0231	{7,6,5;1,2,3}	<64,35,18,20>		Folded 7
<66,10>		66	10	{10,242/27,11/5; 1,55/27,44/5}	1,10,44,11	1,30,20,15	-	-		<66,20,10,4>	Y	M11, Witt 4
<84,8>	!	84	8	{8,45/7,64/15; 1,40/21,12/5}	1,8,27,48	1,18,45,20	-	0123	{18,10,4;1,4,9}	-		Johnson
<91,12>	?	91	12	{12,338/35,39/25; 1,312/175,39/5}	1,12,65,13	1,20,30,40	-	-		-		Van Dam open
<96,19>	-	96	19	{19,12,5; 1,4,15}	1,19,57,19	1,19,57,19	-	0123	{19,12,5;1,4,15}	<96,57,36,30>		2 SRG, Cools [C]
<99,14>	?	99	14	{14,108/11,15/4; 1,24/11,45/4}	1,14,63,21	1,28,42,28	-	-		<99,42,21,15>		Van Dam open

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It has the following parameters:

$$(p^r_{1c}) := \begin{pmatrix} 0 & 20 & 0 & 0 \\ 1 & 3 & 12 & 4 \\ 0 & 8 & 4 & 8 \\ 0 & 2 & 6 & 12 \end{pmatrix} \quad (p^r_{2c}) := \begin{pmatrix} 0 & 0 & 30 & 0 \\ 0 & 12 & 6 & 12 \\ 1 & 4 & 13 & 12 \\ 0 & 6 & 9 & 15 \end{pmatrix}$$

$$(p^r_{3c}) := \begin{pmatrix} 0 & 0 & 0 & 40 \\ 0 & 4 & 12 & 24 \\ 0 & 8 & 12 & 20 \\ 1 & 12 & 15 & 12 \end{pmatrix}$$

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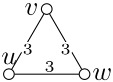
$$(q^r_{1c}) := \begin{pmatrix} 0 & 12 & 0 & 0 \\ 1 & 47/35 & 338/35 & 0 \\ 0 & 312/175 & 303/35 & 39/25 \\ 0 & 0 & 39/5 & 21/5 \end{pmatrix}; \quad q^3_{11} = q^1_{31} = q^1_{13} = 0$$

No integral solutions of the equations with respect to 

`sage-drg` by Janoš Vidali

Using symbolic computation
to prove nonexistence of distance-regular graphs

Janoš Vidali

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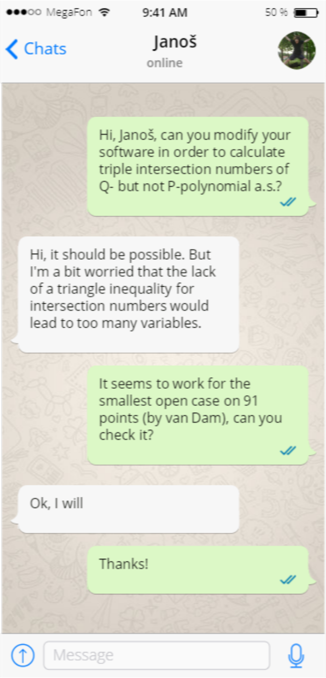
Mathematics Subject Classification: 05E30

Abstract

A package for the Sage computer algebra system is developed for checking feasibility of a given intersection array for a distance-regular graph. We use this tool to show that there is no distance-regular graph with intersection array

$$\{(2r+1)(4r+1)(4t-1), 8r(4rt-r+2t), (r+t)(4r+1); \quad 1, (r+t)(4r+1), 4r(2r+1)(4t-1)\} \quad (r, t \geq 1),$$

$\{135,128,16;1,16,120\}$, $\{234,165,12;1,30,198\}$ or $\{55,54,50,35,10;1,5,20,45,55\}$. In all cases, the proofs rely on the equality in the Krein condition, from which triple intersection numbers are determined. Further combinatorial arguments are then used to derive nonexistence.


●●●○○ MegaFon
9:41 AM
50 %
Chats
Janoš
online
Hi, Janoš, can you modify your software in order to calculate triple intersection numbers of Q- but not P-polynomial a.s.?
Hi, it should be possible. But I'm a bit worried that the lack of a triangle inequality for intersection numbers would lead to too many variables.
It seems to work for the smallest open case on 91 points (by van Dam), can you check it?
Ok, I will
Thanks!
Message

Our results

From Williford's tables:

- 3-class primitive Q-polynomial a.s. on up to 2800 vertices (62 known examples, 359 open cases):
  we ruled out 31 cases, of which 8 correspond to DRGs.
- 4-class Q-bipartite, but not Q-antipodal a.s. on up to 10000 vertices (19 known examples, 488 open cases):
  we ruled out 92 cases.
- 5-class Q-bipartite, but not Q-antipodal a.s. on up to 50000 vertices (7 known examples, 16 open cases):
  we ruled out 12 cases, of which 1 corresponds to a DRG.

Some of the nonexistence results can be extended to infinite families of Q-polynomial a.s. with the following Krein arrays:

- $\{2r^2 - 1, 2r^2 - 2, r^2 + 1; 1, 2, r^2 - 1\}$ (r odd);
- $\{r^3, r^3 - 1, r^3 - r, r^3 - r^2 + 1; 1, r, r^2 - 1, r^3\}$ (r odd);
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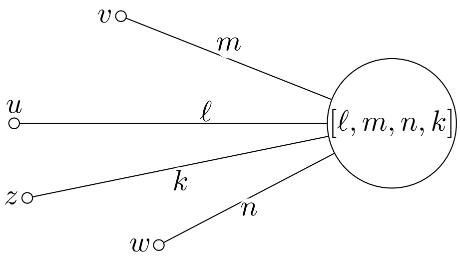
Quadruple intersection numbers

Sho Suda suggested to look at **quadruple** intersection numbers:

$$[\ell, m, n, k]_{u,v,w,z}$$



of **triply regular** association schemes such as:

- some Taylor graphs (= spherical tight 5-design);
- some Q-bipartite Q-polynomial schemes with 4 classes (spherical tight 7-designs);
- some Q-antipodal Q-polynomial schemes with 3-classes with $q_{11}^1 = 0$ (= linked systems of symmetric designs).

Quadruple intersection numbers

Theorem

Let $(X, \mathcal{R})$ be an association scheme with Krein parameters q_{ij}^k $(0 \leq i, j, k \leq D)$. Then

$$\sum_{\ell=0}^{D} \mathrm{Rank}(\mathsf{E}_\ell) q_{ij}^\ell q_{hk}^\ell = 0$$

holds for some indices i, j, h, k if and only if

$$\sum_{r,s,t,p=0}^{D} Q_{ir} Q_{js} Q_{ht} Q_{kp} [r, s, t, p]_{u,v,w,z} = 0$$

for any quadruple $u, v, w, z \in X$.

This condition is satisfied when a scheme is Q-bipartite and $i + j$ and $h + k$ have different parity, however, we didn't find any example where the systems of equations w.r.t. to quadruple intersection numbers would not have an integral solution.

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Further results: tight 4-designs in $H(n,q)$

Suda also suggested to look at some Q-antipodal a.s. (which are not in Williford's tables).

An **orthogonal array** $OA(N,n,q,t)$ is an $N \times n$ matrix M with entries $1,2,\ldots,q$ such that in any $N \times t$ submatrix of M all possible row vectors of length t occur equally often.

Given t (the strength of OA), we are interested in the smallest possible number N.

For $t = 2e$, the lower bound on N was given by Rao (1947):

$$N \geq \sum_{k=0}^{e} \binom{n}{k} (q-1)^k.$$

An orthogonal array is said to be **tight** if it attains this bound.

For $e \geq 3$, $q \geq 3$, there are no tight OA's (Hong, 1986).

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Theorem (Noda, 1979)

If a tight $OA(N, n, q, 4)$ exists, then one of the following holds:

- $(N, n, q) = (16, 5, 2)$;
- $(N, n, q) = (243, 11, 3)$;
- $(N, n, q) = (\frac{9}{2}a^2(9a^2 - 1), \frac{1}{5}(9a^2 + 1), 6)$ where $a \equiv 0 (\mathrm{mod}\ 3)$, $a \equiv \pm 1 (\mathrm{mod}\ 5)$, and $a \equiv 5 (\mathrm{mod}\ 16)$.

(For the first two cases, there exist unique OA's.)

A tight $OA(N, n, q, 4)$

$\Downarrow$

A tight 4-design in $H(n, q)$

$\Downarrow$

A 4-class Q-antipodal a.s.

Its existence is ruled out by using triple intersection numbers.

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$$\begin{array}{c}\text{A tight } OA(N,n,q,4)\\ \Downarrow \\ \text{A tight 4-design in } H(n,q) \\ \Downarrow \\ \text{A 4-class } Q\text{-antipodal a.s.}\end{array}$$

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Thank you!