Combinatorial curvature for infinite planar graphs

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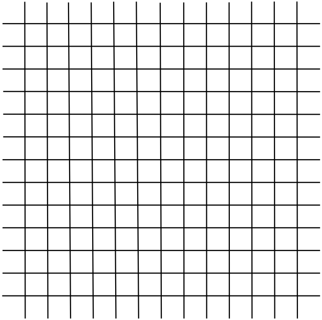
Planar tilings



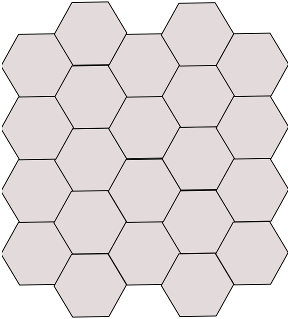


Figure: Square tiling Hexagon tiling

This talk is based on joint works with

- Yohji Akama
  (Tohoku University, Japan).
- Jürgen Jost
  (Max Planck Institute for Mathematics in the Sciences, Germany).
- Shiping Liu
  (University of Science and Technology of China, China).
- Yanhui Su
  (Fuzhou University, China).

- H., J. Jost, and S. Liu.
  Geometric analysis aspects of infinite semiplanar graphs with nonnegative curvature.
  J. Reine Angew. Math., 700:1-36, 2015.
- H. and Y. Su.
  Total curvature of planar graphs with nonnegative curvature.
  arXiv:1703.04119, 2017.
- H. and Y. Su.
  The set of vertices with positive curvature in a planar graph with nonnegative curvature.
  arXiv:1801.02968, 2018.
- Y. Akama, H. and Y. Su.
  Areas of spherical polyhedral surfaces with regular faces.
  arXiv:1804.11033, 2018.

Outline

Planar tessellations

A **planar tessellation** is a planar graph $G = (V, E, F)$ satisfying:

(i) Every face is homeomorphic to a disk whose boundary consists of finitely many edges.

(ii) Every edge is contained in exactly two different faces.

(iii) The intersection of two faces is a vertex or an edge, if it is non-empty.

We only consider planar tessellations satisfying

$$3 \leq \deg(v) < \infty, \quad 3 \leq \deg(\sigma) < \infty, \quad \forall v \in V, \sigma \in F,$$

and call them planar graphs for simplicity.

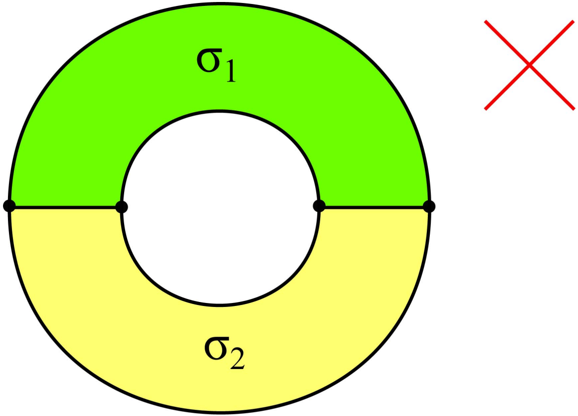
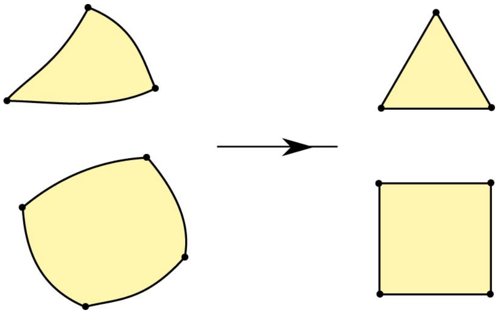


Figure: This violates the condition (*iii*).

Polyhedral surfaces

- For a planar graph G, we endow the ambient space S^2 or $\mathbb{R}^2$ with a metric structure and call it the (regular) **polyhedral surface** of G, denoted by $S(G)$:
  - Assign **each edge length one.**
  - Fill in **each face a regular polygon**.
  - Glue faces along common edges.



Examples

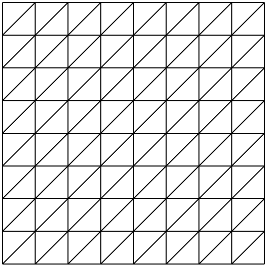
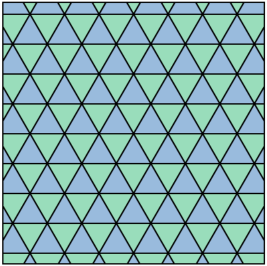


Figure: G



$S(G)$ (*Taken from Wikipedia*)

Examples

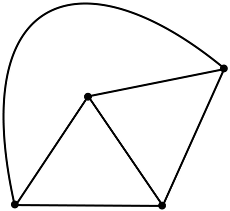
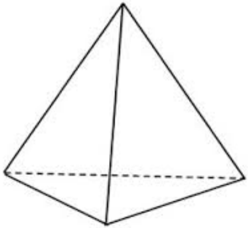
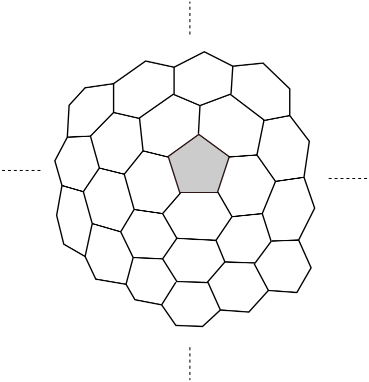


Figure: G



$S(G)$, Tetrahedron

A graph constructed in [Chen-Chen 08] and Réti



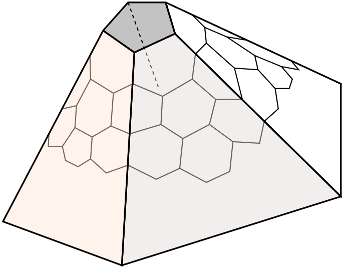


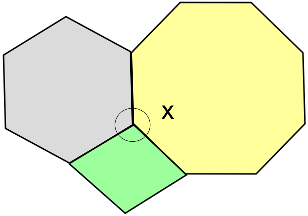
Figure: G $S(G)$ is isometrically embedded in $\mathbb{R}^3$

Generalized Gaussian curvature

The generalized **Gaussian curvature**, i.e. *angle defect*, at a vertex x is defined as

$$K_x := 2\pi - \Sigma_x,$$

where Σ_x is the total angle at x on $S(G)$.



Combinatorial curvature

Definition (Nevanlinna, Stone, Gromov, Ishida, et al)

For a planar graph $G = (V, E, F)$, the combinatorial curvature at $x \in V$ is defined as

$$\Phi(x) := \frac{1}{2\pi} K_x.$$

One can easily show

$$\Phi(x) = 1 - \frac{\deg(x)}{2} + \sum_{\text{face } \sigma \ni x} \frac{1}{\deg(\sigma)}.$$

Negative curvature

Isoperimetric inequalities hold on planar graphs with negative curvature, see e.g. [Żuk97,Woess98,Higuchi01,Häggström-Jonasson-Lyons02, Lawrencenko-Plummer-Zha02, Higuchi-Shirai03,Keller-Peyerimhoff10,Keller11, $\cdots$] .

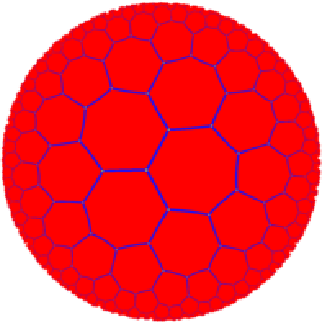
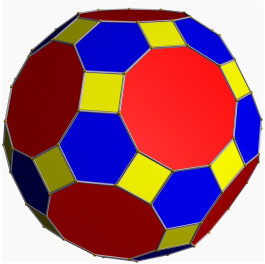


Figure: A uniform tiling of hyperbolic plane, taken from Wikipedia

Examples of planar graphs with positive curvature

Polyhedra with regular faces in $\mathbb{R}^3$ are classified as follows:

1. 5 Planotic solids,
2. 13 Archimedean solids,
3. 92 Johnson solids, [Johnson66, Zalgaller67],
4. prisms and antiprisms.



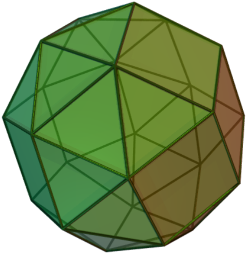


Figure: Great rhombicosidodecahedron, Gyroelongated square bicupola (J45), taken from Wikipedia

Prisms and antiprisms

Prisms and antiprisms are shown as below:

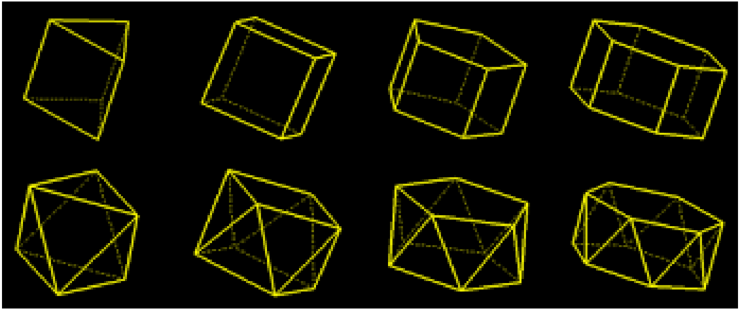


Figure: Cite: http://maths.ac-noumea.nc/polyhedr/convexA_.htm

Planar graphs with positive curvature

- [Stone76] proved that a planar graph $G = (V, E, F)$ is finite if $\Phi(x) \geq c > 0, \ \forall x \in V$.
- [Higuchi01] conjectured that any planar graph with positive curvature is finite.

Theorem (DeVos-Mohar '07, Chen-Chen '08)

Let $G = (V, E, F)$ be a planar graph with positive curvature. Then G is a finite graph. Moreover, if it is not a prism or an antiprism, then

$$\sharp V \leq 3444.$$

Planar graphs with nonnegative curvature

Denote by

$$\mathcal{PC}_{\geq 0} := \{G : \Phi(x) \geq 0,\ x \in V\}$$

the class of planar graphs with nonnegative curvature.
In fact,

$$G \in \mathcal{PC}_{\geq 0} \Longleftrightarrow S(G) \textit{ is a convex surface } (\textit{Alexandrov}).$$

Real world applications

A *fullerene* graph is a 3-regular finite planar graph consisting of pentagons and hexagons as faces.

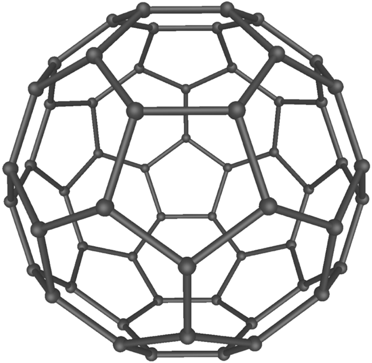


Figure: Buckminsterfullerene, C_{60}, from Wikipedia

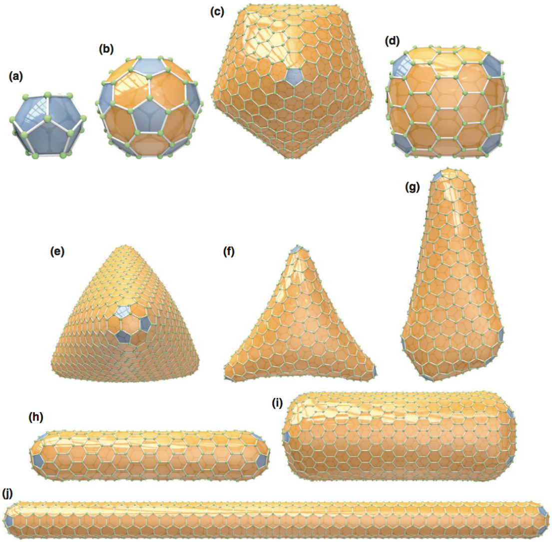


Figure: Fullerenes, taken from Schwerdtfeger-Wirz-Avery, The topology of fullerenes, WIREs Comput Mol Sci 2015, 5:96-145.

Combinatorial complexity of fullerenes

- Given the number of hexagons k, denote by $f(k)$ the number of non-isomorphic types of fullerenes.

The enumeration was obtained by using computer [Brinkmann-Dress '97].[1]

k	$f(k)$	k	$f(k)$	k	$f(k)$
0	1	30	31924	60	7341204
1	0	31	39718	61	8339033
2	1	32	51592	62	9604410
3	1	33	63761	63	10867629
4	2	34	81738	64	12469092
5	3	35	99918	65	14059173

[1] A Constructive Enumeration of Fullerenes, J. Algorithms 23(2), 345-358, 1997.

A theoretic result

[Thurston '98][2] proved that there are $c, C > 0$, such that

$$c\,k^9 \le f(k) \le C\,k^9, \quad \forall k \ge 1.$$

[2] Shapes of polyhedra and triangulations of the sphere, The Epstein birthday schrift, Geom. Topol. Monogr., vol. 1, Geom. Topol. Publ., Coventry 1998, 511-549.

Tilings with regular polygons-I

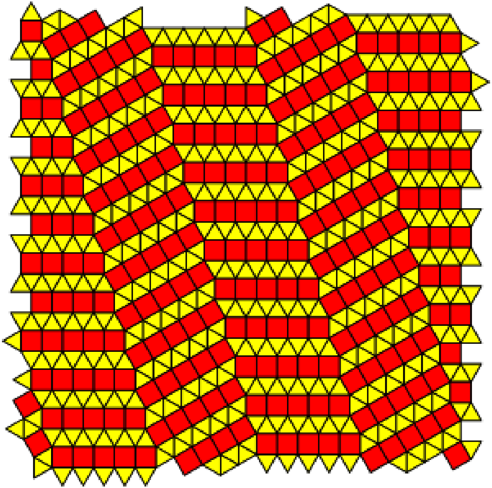


Figure: Downloaded from Wikipedia.

Tilings with regular polygons-II

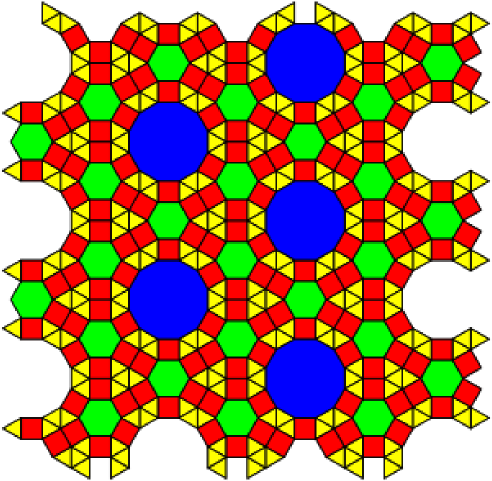


Figure: Downloaded from Wikipedia.

Analysis on infinite graphs in $\mathcal{PC}_{\geq 0}$

Theorem (H.-Jost-Liu '15)

*Any $G \in \mathcal{PC}_{\geq 0}$ has quadratic volume growth. Moreover, the **volume doubling** and the **Poincaré inequality** hold.*

1. Quadratic volume growth $\Longrightarrow$ Recurrent simple random walk.
2. Volume doubling+Poincaré inequality $\Longrightarrow$
   - Harnack inequality [Grigor'yan, Saloff-Coste, Delmotte].
   - Yau's conjecture on polynomial growth harmonic functions [Colding-Minicozzi, Li, Kleiner, Delmotte, H.-Jost-Liu] .

General questions

Question

How to **construct** *a planar graph with nonnegative curvature?*

Question

What are the **obstructions** *to construct such a graph?*

Outline

Theorem A (Total curvature)

For any planar graph G, we denote by

$$\Phi(G) := \sum_{x \in V} \Phi(x)$$

the total curvature of G.

Theorem (H.-Su '17)

$$\{\Phi(G) : \text{ infinite graph } G \in \mathcal{PC}_{\geq 0}\}$$
$$= \left\{\frac{k}{12} : 0 \leq k \leq 12, k \in \mathbb{Z}\right\}.$$

Theorem B (The set of vertices with positive curvature)

For any $G \in \mathcal{PC}_{\geq 0}$, we denote by

$$T_G := \{v \in V : \Phi(x) > 0\}$$

the set of vertices with positive curvature.

Theorem (H.-Su '18)

$$\max_G \sharp T_G = 132,$$

where the maximum is taken over all infinite graphs $G \in \mathcal{PC}_{\geq 0}$ which are not "prism-like".

An application

Corollary

For an infinite graph $G \in \mathcal{PC}_{\geq 0}$, *if it is not "prism-like" and* $\Phi(G) > 0$, *then*

$$\sharp \mathrm{Aut}(G) \mid 132! \times 120,$$

where $\mathrm{Aut}(G)$ *is the cellular automorphism group of G. In particular,*

$$\mathrm{Aut}(G) \leq 10^{227}.$$

Outline

Euler characteristic

Theorem (Euler's formula)

For a graph $G = (V, E, F)$ embedded into a closed surface M,

$$\sharp V - \sharp E + \sharp F = \chi(M),$$

where $\chi(\cdot)$ denotes the Euler characteristic of a surface.

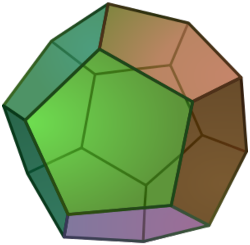


Figure: Dodecahedron (Taken from Wikipedia)

Gauss-Bonnet theorem-compact case

From now on, we denote by

$$K(M) := \int_M K(x) d\mathrm{vol}(x)$$

the total curvature of a smooth surface M.

Theorem (Gauss-Bonnet)

For a closed surface M,

$$K(M) = 2\pi\chi(M).$$

Theorem (Discrete Gauss-Bonnet)

For a finite polyhedral surface $S(G)$,

$$K(G) = 2\pi\chi(S(G)).$$

Noncompact case

Theorem (Huber '57, Cohn-Vossen '59)

For a complete surface M, *if* $|\int_M K_-(x)dx| < \infty$, *then*

$$K(M) \leq 2\pi\chi(M).$$

Theorem (DeVos-Mohar '07, Chen-Chen '08)

For an infinite graph $G \in \mathcal{PC}_{\geq 0}$,

$$\Phi(G) \leq 1.$$

Geometric meaning of total curvature

Theorem

Let M be a convex surface and $C(X)$ be the cone at infinity of M, where X is a circle S^1. Then the total curvature

$$K(M) = 2\pi - \text{length}(X).$$

See Shiohama, Shioya, and Tanaka. [3]

This yields

$$\{K(M) : \text{ convex surface } M\} = [0, 2\pi].$$

[3] "The geometry of total curvature on complete open surfaces," Cambridge University Press, 2003.

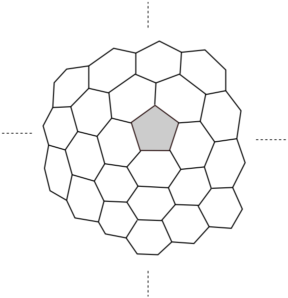
Réti's question

Question (Réti)

What is the number

$$\min\{\Phi(G) : G \in \mathcal{PC}_{\geq 0}, \Phi(G) > 0\}?$$

The minimum was conjectured to be obtained by the following graph G with $\Phi(G) = \frac{1}{6}$.



Theorem A

Theorem (H.-Su '17)

$$\min\{\Phi(G) : G \in \mathcal{PC}_{\geq 0}, \Phi(G) > 0\} = \frac{1}{12}.$$

Theorem (H.-Su '17)

$$\{\Phi(G) : \text{ infinite graph } G \in \mathcal{PC}_{\geq 0}\}$$
$$= \left\{\frac{k}{12} : 0 \leq k \leq 12, k \in \mathbb{Z}\right\}.$$

Example I

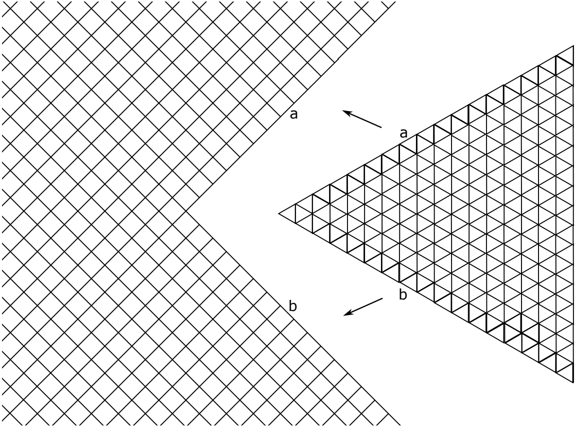


Figure: $\Phi(G) = \frac{1}{12}$.

Example II

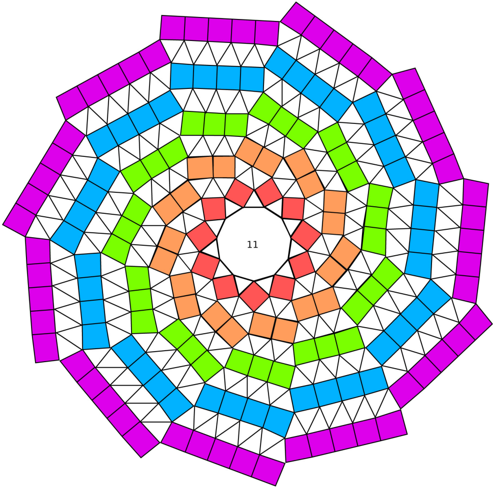


Figure: $\Phi(G) = \frac{1}{12}$.

More examples

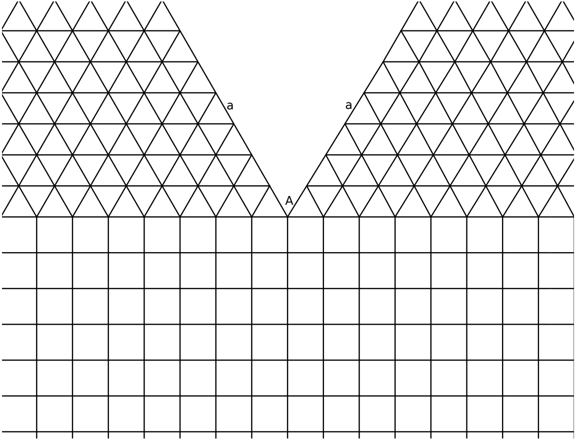


Figure: $\Phi(G) = \frac{1}{6}$.

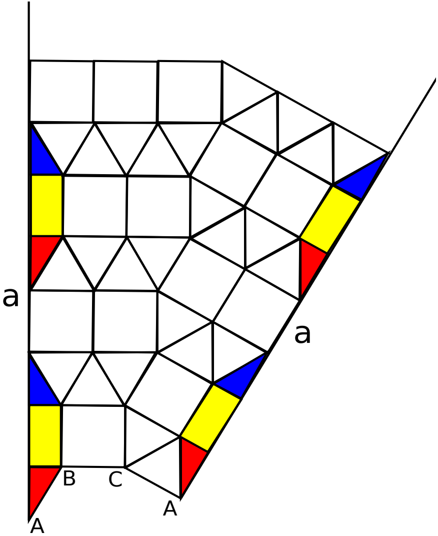


Figure: $\Phi(G) = \frac{11}{12}$.

The proof of Theorem A: two ingredients

- The Gauss-Bonnet theorem on **surfaces with boundary**: For a compact polyhedral domain Ω,

$$2\pi \sum_{x \in V \cap \mathrm{int}(\Omega)} \Phi(x) + \sum_{y \in V \cap \partial\Omega} (\pi - \theta_y) = 2\pi\chi(\Omega),$$

  where θ_y is the inner angle of a vertex $y \in \partial\Omega$.
- Chen-Chen's theorem.

Theorem (Chen-Chen)

For any $G \in \mathcal{PC}_{\geq 0}$,

$$\sharp T_G < \infty.$$

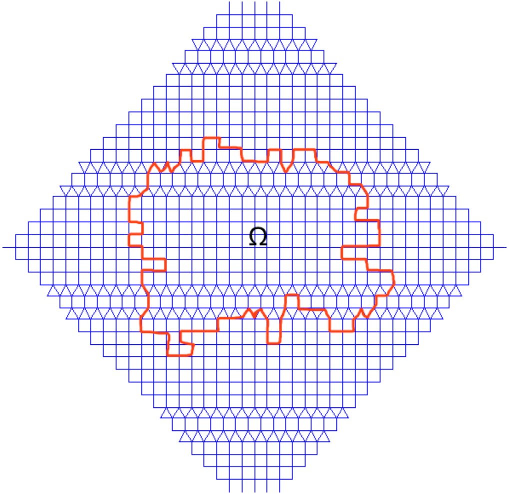


Figure: There are only triangles and squares outside Ω. Figure is modified from B. Galebach, n-Uniform tilings. http://probabilitysports.com/tilings.html.

The pattern of vertices with vanishing curvature

A pattern of a vertex x consists of all faces contains x, denoted by $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ where $x \in \sigma_i$.

$(3,7,42)$,	$(3,8,24)$,	$(3,9,18)$,	$(3,10,15)$,	$(3,12,12)$,
$(4,5,20)$,	$(4,6,12)$,	$(4,8,8)$,	$(5,5,10)$,	$(6,6,6)$,
$(3,3,4,12)$,	$(3,3,6,6)$,	$(3,4,4,6)$,	$(4,4,4,4)$,	$(3,3,3,3,6)$,
$(3,3,3,4,4)$,	$(3,3,3,3,3,3)$.			

Table 2. The patterns of a vertex with vanishing curvature.

Lemma

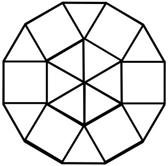
For any vertex x satisfying $\Phi(y) = 0$ for any $y \in B_2(x)$, the possible patterns of x are in the following:

$$(3,12,12), (4,6,12), (4,8,8), (6,6,6), (3,3,4,12), (3,3,6,6),$$

$$(3,4,4,6), (4,4,4,4), (3,3,3,3,6), (3,3,3,4,4), (3,3,3,3,3,3).$$

Divide

- any hexagon into 6 triangles, and
- any dodecagon into 12 triangles and 6 squares.



Then any face outside a (large) compact polyhedral domain Ω is a triangle, a square or an octagon.

Moreover, by the connectedness argument, we have two cases:

1. All faces outside Ω are either triangles or squares.
2. All vertices outside Ω are of pattern $(4, 8, 8)$.

Then one is ready to show

$$\sum_{y \in V \cap \partial\Omega} (\pi - \theta_y) = k \cdot \frac{\pi}{6}, \quad \text{for some } k \in \mathbb{Z}.$$

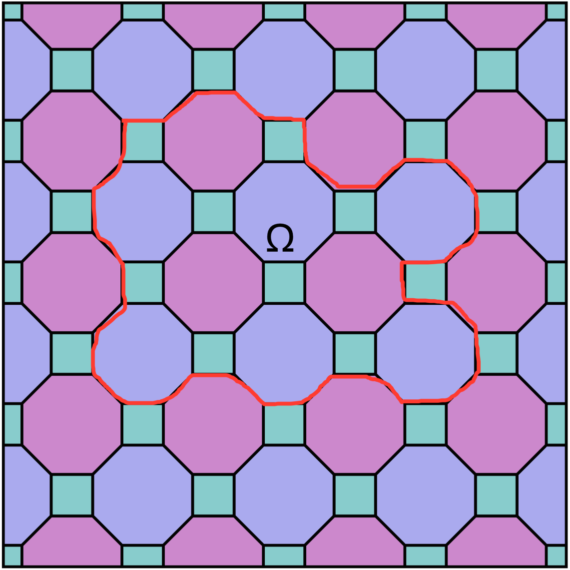


Figure: There are only squares and octagons outside Ω.

Outline

Planar graphs with positive curvature

Question (DeVos-Mohar '07)

What is

$$C_{\mathbb{S}^2} := \max_G \sharp V,$$

where the maximum is taken over all planar graphs with positive curvature which are not prisms or antiprisms?

[Nicholson-Sneddon '11, Oldridge '17+, Ghidelli '17] constructed examples to show that

$$C_{\mathbb{S}^2} \geq 208.$$

By the discharging method, [Oh '17] proved that $C_{\mathbb{S}^2} \leq 380$.

Theorem (Ghidelli '17)

$C_{\mathbb{S}^2} = 208.$

An example with 208 vertices

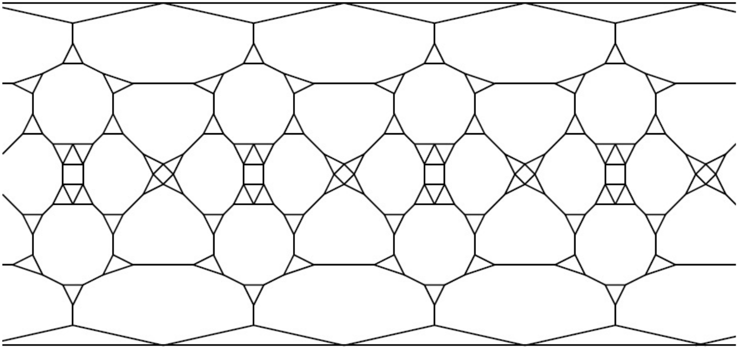


Figure: Taken from [Nicholson-Sneddon '11]

Prism-like graphs

Definition

An infinite graph $G \in \mathcal{PC}_{\geq 0}$ is called a **prism-like graph** if

$$\sup_{\sigma \in F} \deg(\sigma) \geq 43.$$

Theorem (H.-Jost-Liu '15)

For a prism-like infinite graph $G = (V, E, F)$ with no hexagonal faces,

$$F = \sigma \cup (\cup_{i=1}^{\infty} L_i),$$

where L_i, $i \geq 1$, is a set of faces of the same type, triangle or square, which composite a band and $\deg(\sigma) \geq 43$. Moreover, $S(G)$ is isometric to the boundary of a half flat-cylinder in $\mathbb{R}^3$.

A prism-like graph

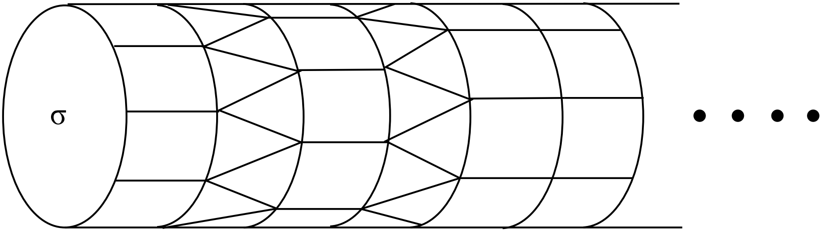


Figure: A half flat-cylinder in $\mathbb{R}^3$.

Theorem B

Theorem (H.-Su '18)

$$\max_G \sharp T_G = 132,$$

where the maximum is taken over all infinite graphs $G \in \mathcal{PC}_{\geq 0}$ which are not prism-like graphs.

Discharging methods

Aim: For any $G \in \mathcal{PC}_{\geq 0}$, not prism-like,

$$\sharp T_G \leq 132.$$

Discharging process $\Phi \to \widetilde{\Phi}$ is given as follows:
Redistribute the curvature function $\Phi : V \to \mathbb{R}_{\geq 0}$ to get a modified curvature function $\widetilde{\Phi} : V \to \mathbb{R}_{\geq 0}$ such that

1. $\sum_{x \in V} \widetilde{\Phi}(x) = \sum_{x \in V} \Phi(x)$, and
2. $\widetilde{\Phi}(x) \geq \frac{1}{132}, \quad \forall x \in T_G.$

The result follows from

$$\frac{1}{132} \sharp T_G \leq \sum_{x \in T_G} \widetilde{\Phi}(x) \leq 1.$$

A vertex in G is called **bad** if $0 < \Phi(x) < \frac{1}{132}$, otherwise it is called good. We shall transfer the curvature on good vertices to bad vertices, as required.
The list of vertex patterns for bad vertices.

$(3,7,k)$, $32 \le k \le 41$	$(3,8,k)$, $21 \le k \le 23$
$(3,9,k)$, $k = 16, 17$	$(3,10,14)$,
$(3,11,13)$,	$(4,5,k)$, $k = 18, 19$
$(4,7,9)$,	

One case in the proof.

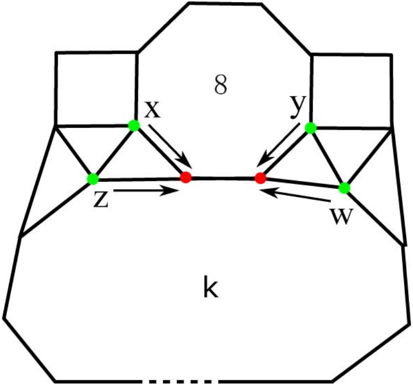
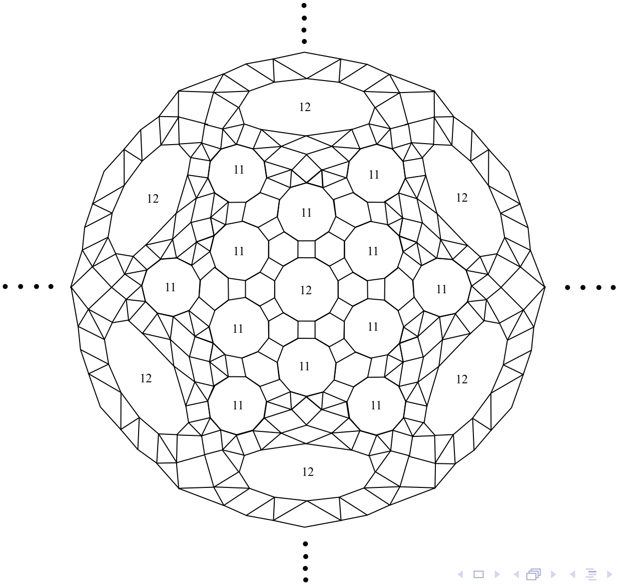


Figure: Distribute the curvature on good vertices, x, y, z, w, to bad vertices on k-gon.

An example with $\sharp T_G$=132.



The example can be extended to the infinity.

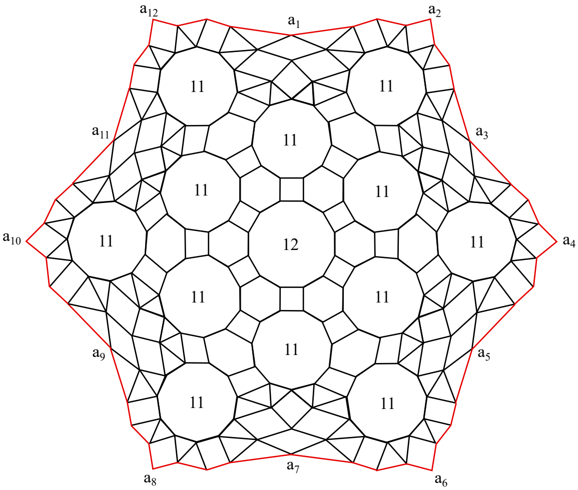


Figure: A central part in the example, denoted by P_A.

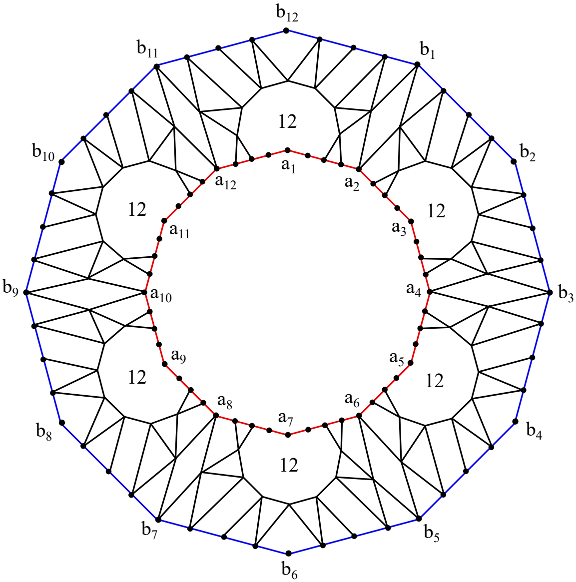
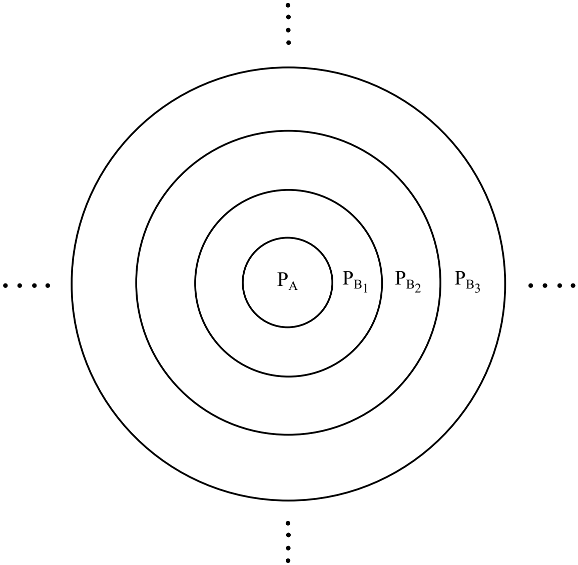


Figure: An annular part in the example, denoted by P_B.

A “periodic" structure in the example



Outline

Spherical polyhedral surfaces

- For $a > 0$, we endow the ambient space of G with a metric structure, called the (regular) **spherical polyhedral surface** of G and denoted by $S_a(G)$:
  - Fill in each face a regular **spherical polygon** in the unit sphere $\mathbb{S}^2$ of side length a.
  - Glue faces along common edges.

The **Gaussian curvature** at the vertex x is defined as

$$K_a(x) := 2\pi - \Sigma_a(x),$$

where $\Sigma_a(x)$ is the total angle at x.

Almost spherical tilings

Denote by

$$\mathcal{PC}_{\geq 1} := \{G : \text{ there exists } a > 0 \text{ such that } S_a(G) \in \mathrm{Alex}(1)\}$$

the **almost spherical tilings,** where $\mathrm{Alex}(1)$ denotes the space of Alexandrov space with curvature at least 1. We can show that

$$\mathcal{PC}_{\geq 1} = \{G : \Phi(x) > 0, \ \forall x \in V\}.$$

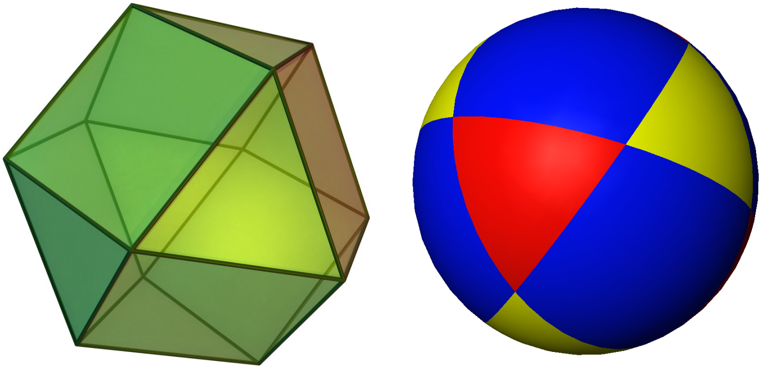


Figure: Downloaded from Wikipedia.

Theorem (Akama-H.-Su '18)

The spherical tilings with regular faces are:

1. *5 Platonic solids,*
2. *13 Archimedean solids,*
3. *prisms and antiprisms,*
4. *22 Johnson solids J*1*, J*3*, J*6*, J*11*, J*19*, J*27*, J*34*, J*37*, J*62*, J*63*, J*72*, J*73*, J*74*, J*75*, J*76*, J*77*, J*78*, J*79*, J*80*, J*81*, J*82*, J*83*.*

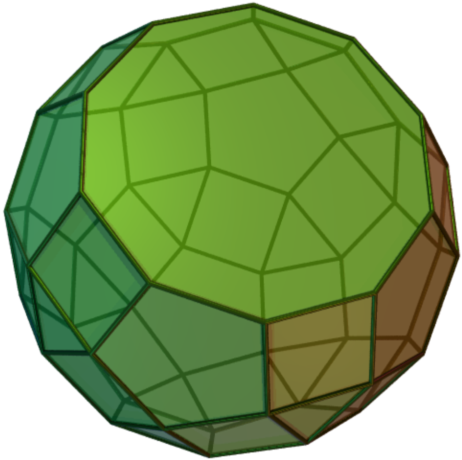


Figure: Downloaded from Wikipedia.

Theorem C (Total area of almost spherical tilings)

Theorem (Akama-H.-Su '18)

Let G be an almost spherical tiling, which is not a spherical tiling. Then for any $a > 0$ such that $S_a(G) \in \mathrm{Alex}(1)$,

$$\mathrm{Area}(S_a(G)) \leq 4\pi - 1.6471 \times 10^{-5}.$$

The ingredient for the proof

Theorem (Discrete Gauss-Bonnet Theorem)

For a finite spherical polyhedral surface $S_a(G)$, $a > 0$,

$$\mathrm{Area}(S_a(G)) + \sum_{x \in V} K_a(x) = 4\pi.$$

Open problems

Question

Refine the order estimate of the cellular automorphism group for G, $G \in \mathcal{PC}_{\geq 0}$ and $\Phi(G) > 0$.

Question

Is there any infinite graph $G \in \mathcal{PC}_{\geq 0}$ which has a vertex of pattern $(3, 7, 41)$?

Question

What is the number

$$\max \mathrm{Area}(S_a(G)),$$

where the maximum is taken over all planar graphs G, which are not spherical tiling, and $a > 0$ such that $S_a(G) \in \mathrm{Alex}(1)$?

Thank You for Your Attention.

Patterns		$\Phi(x)$
$(3,3,k)$	$3 \leq k$	$1/6 + 1/k$
$(3,4,k)$	$4 \leq k$	$1/12 + 1/k$
$(3,5,k)$	$5 \leq k$	$1/30 + 1/k$
$(3,6,k)$	$6 \leq k$	$1/k$
$(3,7,k)$	$7 \leq k \leq 41$	$1/k - 1/42$
$(3,8,k)$	$8 \leq k \leq 23$	$1/k - 1/24$
$(3,9,k)$	$9 \leq k \leq 17$	$1/k - 1/18$
$(3,10,k)$	$10 \leq k \leq 14$	$1/k - 1/15$
$(3,11,k)$	$11 \leq k \leq 13$	$1/k - 5/66$
$(4,4,k)$	$4 \leq k$	$1/k$
$(4,5,k)$	$5 \leq k \leq 19$	$1/k - 1/20$
$(4,6,k)$	$6 \leq k \leq 11$	$1/k - 1/12$
$(4,7,k)$	$7 \leq k \leq 9$	$1/k - 3/28$
$(5,5,k)$	$5 \leq k \leq 9$	$1/k - 1/10$
$(5,6,k)$	$6 \leq k \leq 7$	$1/k - 2/15$
$(3,3,3,k)$	$3 \leq k$	$1/k$
$(3,3,4,k)$	$4 \leq k \leq 11$	$1/k - 1/12$
$(3,3,5,k)$	$5 \leq k \leq 7$	$1/k - 2/15$
$(3,4,4,k)$	$4 \leq k \leq 5$	$1/k - 1/6$
$(3,3,3,3,k)$	$3 \leq k \leq 5$	$1/k - 1/6$

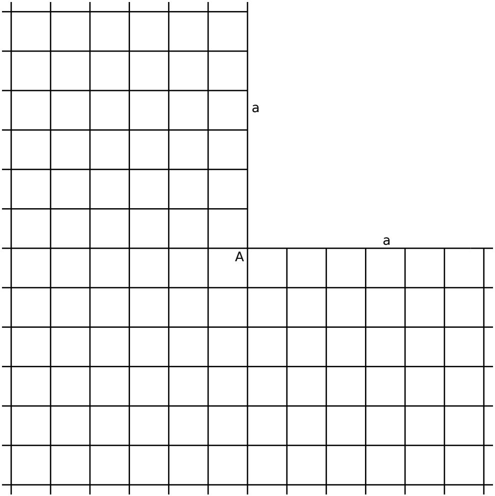


Figure: A graph with total curvature $\frac{3}{12} \cdot 2\pi$.

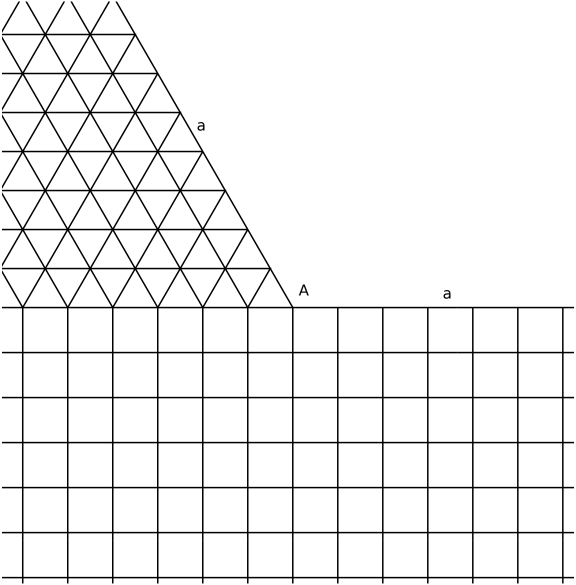


Figure: A graph with total curvature $\frac{4}{12} \cdot 2\pi$.

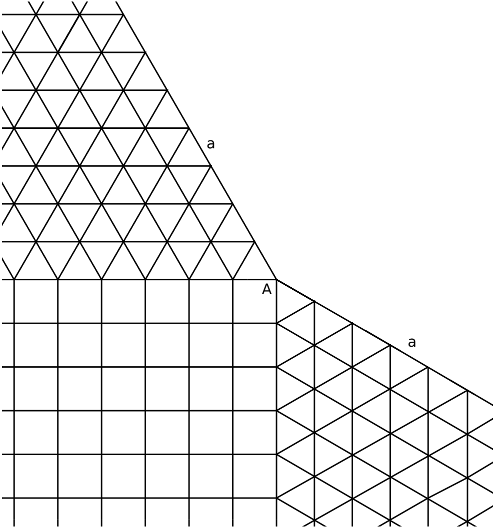


Figure: A graph with total curvature $\frac{5}{12} \cdot 2\pi$.

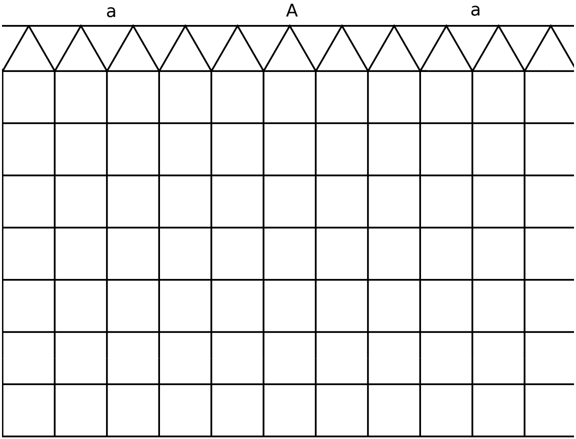


Figure: A graph with total curvature $\frac{6}{12} \cdot 2\pi$.

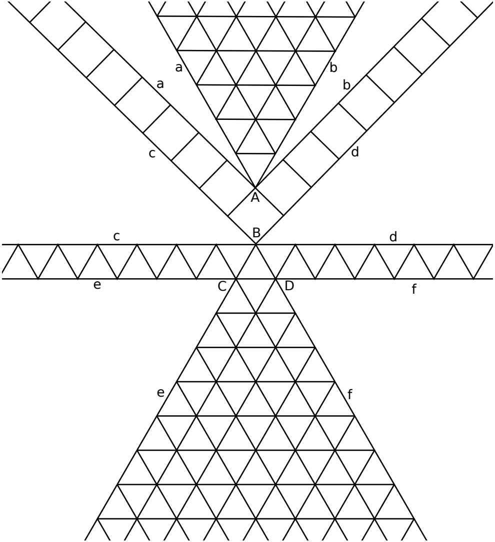


Figure: A graph with total curvature $\frac{8}{12} \cdot 2\pi$.

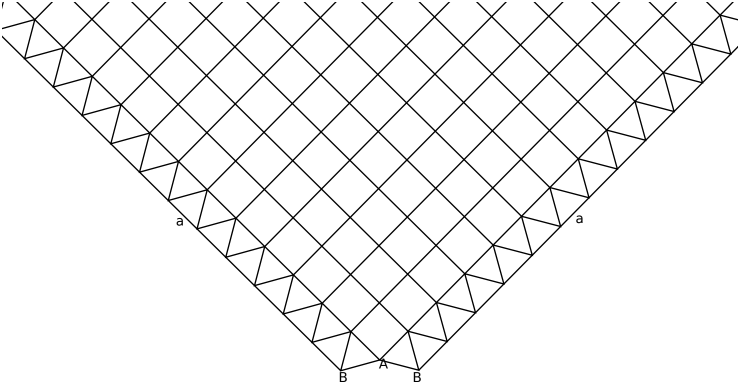


Figure: A graph with total curvature $\frac{9}{12} \cdot 2\pi$.

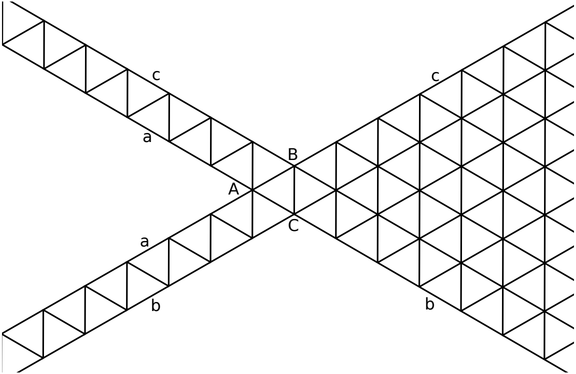


Figure: A graph with total curvature $\frac{10}{12} \cdot 2\pi$.

Volume doubling and Poincaré inequality

Definition

We say a graph $G = (V, E)$ satisfies

- the **volume doubling** if

$$\sharp B_{2R}(p) \leq C_1 \sharp B_R(p), \quad \forall\, p \in V, R > 0;$$

- the **Poincaré inequality** if for any function u, $p \in V$, $R > 0$

$$\inf_a \sum_{x \in B_R(p)} |u(x) - a|^2 \leq C_2 R^2 \sum_{x,y \in B_{2R}(p)} (u(x) - u(y))^2.$$

Alexandrov conjecture: For a closed oriented convex surface M,

$$\frac{\mathrm{diam}^2(M)}{\mathrm{Area}(M)} \geq \frac{2}{\pi}.$$

The equality holds for the doubly-covered disc. See [Calabi-Cao '92, Shioya '15] for recent results.
For any fullerene $G = (V, E, F)$, Andova et al. '12 proved

$$\mathrm{diam}(G) \geq \sqrt{\frac{2}{3}\sharp V - \frac{5}{18}} - \frac{1}{2}.$$

Question (Diameter of fullerenes)

What is the sharp constant C such that

$$\mathrm{diam}^2(G) \geq C\,\sharp V \quad \text{holds for sufficiently large fullerene G?}$$