Intersection of conjugate solvable subgroups in $GL(n, q)$

Anton Baykalov

The University of Auckland

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Base and Base size

Let a group G acts on set Ω. A sequence of points $B = (\beta_1, \ldots, \beta_m)$ from Ω is a **base** for G if the pointwise stabilzer of B is equal to the kernel:

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Two bases $(\beta_1, \ldots, \beta_m)$ and $(\beta'_1, \ldots, \beta'_m)$ of size m are **equivalent** if there exist $g \in G$ such that

$$(\beta_1 \cdot g, \ldots, \beta_m \cdot g) = (\beta'_1, \ldots, \beta'_m)$$

Let G acts on Ω transitively. Since for every $\alpha, \beta = (\alpha)g \in \Omega$

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the question about base size can be formulated in the following way:

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the question about base size can be formulated in the following way: Given $H \leqslant G$ find the minimal number $b_H(G)$ of conjugates of H such that their intersection equals $H_G = \cap_{g \in G} H^g$.

Let G act transitively and H is the stabilizer of a point, so $|\Omega| = |G : H|$. If $(\beta_1, \ldots, \beta_m)$ is a base then

$$|(\beta_1, \ldots, \beta_m)^G| \leqslant |\Omega| \cdot (|\Omega| - 1) \ldots (|\Omega| - m + 1) < |\Omega|^m = |G : H|^m.$$

Therefore

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Theorem (Babai, Goodman, Pyber, 1997)

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Theorem (Babai, Goodman, Pyber, 1997)

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They also conjectured that constant c is at most 7.

Problem (Vdovin, "Kourovka notebook", 17.41)

Let S be a maximal solvable subgroup of a finite group G.
Is it true that $b_S(G) \leqslant 5$?

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Example

If $G = S_8$; $S := S_4 \wr S_2$ then $b_S(G) = 5$.

Known results: reduction and symmetric groups

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Theorem 1(B., 2017)

Let H be a solvable subgroup of an almost simple group G with socle isomorphic to A_n, $n \geqslant 5$. Then $Reg_H(G, 5) \geqslant 5$. In particular $b_H(G) \leqslant 5$.

Classical groups

Theorem 2 (Burness,2007)

Let G be a finite almost simple classical group in a faithful primitive non-standard action. Then either $b(G) \leqslant 4$, or $G = U_6(2) \cdot 2$, $H = U_4(3) \cdot 2^2$ and $b(G) = 5$.

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Theorem 3, (B. to appear...)

Let $G = GL(n, q)$, $n \geqslant 2$ and (n, q) is not equal to $(2, 2)$ or $(2, 3)$. If S is maximal solvable subgroup of G then $Reg_S(G, 5) \geqslant 5$, in particular $b_S(G) \leqslant 5$.

Sketch of the part of the proof

Lemma 1

If S is maximal primitive solvable subgroup of $GL(n,q)$ then $b_S(G) \leqslant 2$ with a series of exception for $n \leqslant 4$ (in those cases $b_S(G) \leqslant 3$).

Lemma 2

If S is a maximal irreducible solvable subgroup of $GL(n,q)$ then
1) S is conjugate to $S_1 \wr \Gamma$, where $S_1 \leqslant GL(k,q)$ is primitive; $\Gamma \leqslant Sym(l)$ is transitive; $n = kl$.
2) $b_S(G) \leqslant b_{S_1}(GL(k,q))$.

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$$Q(G,c) \leqslant \sum_{x \in \mathscr{P}} \left(\frac{|C_\Omega(x)|}{|\Omega|} \right)^c = \sum_{i=1}^{k} |x^G| \cdot \left(\frac{|x^G \cap H|}{|x^G|} \right)^c =: \widehat{Q}(G,c).$$