Gröbner-Shirshov bases for groups, semigroupss, categories and Lie algebras

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Early results

Shirshov (1962), Hironaka (1964), Buchberger (1965, 1970).

Some early results (1960-th)

- Theory of one-relator Lie algebras.

- Unsolvability of the word problem for Lie algrebras.

- Novikov and Boon groups, new proofs

- Turing degrees of the conjugacy problem for groups.

- The semigroup algebra imbeddable into a group but not embeddable into any division algebra

Early results

(1) Word problem for any one-relation Lie algebra is algorithmically solvable (Shirshov, 1962)

Freedom Theorem (Magnus Freiheitssatz) is valid for one-relation Lie algebra (Shirshov, 1962)

Any Lie algebra is embeddable into an algebraically closed Lie algebra L: for any non-constant polynomial $f(x_1, ..., x_n) \in L * \mathrm{Lie}(X)$ (the free product), equation $f(x_1, ..., x_n) = 0$ has a solution in L. Here $\mathrm{Lie}(X)$ is a free Lie algebra. (B., 1962, it was a part of my PhD Thesis, 1963)

(2) Some recursively presented Lie algebra with insoluble word problem is effectively embeddable into a finitely presented Lie algebra (with insoluble word problem) (a weak Higman type theorem; it was proved with a help of Matiyasevich Diophantine presentation of any recursively enumerable set of natural numbers) (B., 1972)

In (1), (2), it is used GS bases for Lie algebras

Early results

(3) There are given new fairly brief proofs that two Novikov (1954, 1955) and Boon (1960) groups have insoluble word (conjugacy) problem.

(4) For any Turing degree of insolubility α there exist a first Novikov group (1954) with the same Turing degree of insolubility of the conjugacy problem (Malcev problem) (B., 1968).

(5) It is given a (quadratic 12-relation) semigroup algebra (without zero divisors) embeddable into a group but not embeddable into any division algebra (Malcev problem) (B., 1969). Up to now it is only known semigroup with these properties.

In (3), (4),(5), it is used the notion of groups with (relative) standard bases, a version of GS bases for towers of HNN extention of groups

The results (3)-(5) are in my Dr. Sci. Thesis, 600 pp, 1969.

Composition-Diamond lemma for associative algebras (A.I.Shirshov (implicitly, 1962), L.A.B. (1976), G.Bergman (1978))

$S \subset k\langle X\rangle$, k - field, $(X^*, \leqslant)$ - monomial well ordering
$f = \bar{f} + r_f$, $r_f < \bar{f}$, $f \in S$—monic polynomial

Compositions

$$(f, g)_w = fa - bg \quad (w = \bar{f}a = b\bar{g}, \bar{f} \cap \bar{g} \neq 1)$$

or

$$f - agb \quad (w = \bar{f} = a\bar{g}b)$$

S - GSB:

$$(f, g)_w = \sum \alpha_i a_i s_i b_i, \ \text{ with } a_i\bar{s}_i b_i < w, \ f, g, s_i \in S.$$

CD Lemma for associative algebras. $S \subset k\langle X\rangle$. TFAE

(i) S is a GSB,

(ii) $f \in \mathrm{Ideal}(S) \Longrightarrow \bar{f} = a\bar{s}b, s \in S$,

(iii) $\mathrm{Irr}(S) = \{u \mid u \neq a\bar{s}b, s \in S\}$ is a linear basis of $k\langle X|S\rangle$.

Lie compositions

monic $f, g \in S \subseteq \mathrm{Lie}(X) \subseteq k\langle X\rangle, (X^*, \leqslant deg - lex)$, Lyndon-Shirshov words, Lyndon-Shirshov base, $\bar{f}, \bar{g}$ - maximal ASSOCIATIVE words of f, g, $(f, g)_w$ associative compositions of f, g ($fb - ag$, or $f - agb$)

Lie compositions $\mathrm{Lie}(f, g)_w$ - the special Shirshov bracketing of the associative compositions

$$\mathrm{Lie}(f, g)_w = [fa]_{\bar{f}} - [bg]_{\bar{g}}, \quad f, g \in S \quad (w = \bar{f}a = b\bar{g}, a, b \in X^*)$$

$$\text{or} \qquad f - [agb]_{\bar{g}} \quad (w = \bar{f} = a\bar{g}b)$$

S— Lie GSB:

$$\mathrm{Lie}(f, g)_w = \sum \alpha_i [a_i s_i b_i]_{\bar{s}_i},$$

where $a_i \bar{s}_i b_i < w, f, g, s_i \in S$.

CD-lemma for Lie algebras (A.I.Shirshov, 1962)

$S \subset Lie(X) \subset k\langle X\rangle$. TFAE

(i) S is a Lie GSB,

(ii) $h \in \text{LieIdeal}(S) \Longrightarrow \bar{h} = a\bar{s}b, s \in S$,

(iii) $\text{Irr}(S) = \{[u] \mid u \neq a\bar{s}b, u$ is a Lyndon-Shirshov word, $s \in S\}$ is a linear base of $Lie(X|S)$.

Finite Coxeter groups of type A_l

Coxter group A_l is generated by $s_1, ..., s_l$ with defining relations

$s_i^2 = 1$, $s_i s_j = s_j s_i$, where $i - j > 1$; $s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$, $1 \leqslant i \leqslant l - 1$.

Let us define words $s_{ij} = s_i s_{i-1} ... s_j$, where $i > j$; and $s_{ii} = s_i$, $s_{i,i+1} = 1$. Theorem (L.A.B., L.-S. Shiao, 2001) The reduced Gröbner-Shirshov basis of the Coxter group A_l consists of relations $s_{i+1j} s_{i+1} = s_i s_{i+1j}$, where $1 \leqslant j < i$, together with the initial relaitons of A_l. The corresponding reduced set of words of the form:

$$s_{1j_1} s_{2j_2} ... s_{lj_l},$$

where $1 \leqslant j_i \leqslant i + 1$ for all i.

Corrollary 1. $|A_l| = (l + 1)!$.

Corrolary 2. Coxeter group A_l is isomorphic to the symmetric group S_{l+1} under the isomorphism $s_i \mapsto (i, i + 1)$.

Coxter group B_l is generated by s_i, $1 \leqslant i \leqslant l$, with defining relations

$s_i^2 = 1, s_i s_j = s_j s_i, i - j > 1;$

$s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i, 1 \leqslant i \leqslant l-2;$

$s_l s_{l-1} s_l s_{l-1} = s_{l-1} s_l s_{l-1} s_l.$

Let us define elements s_{ij}, $1 \leqslant j \leqslant i+1$, $i \leqslant l$ as before.

The following relations are valid in B_l:

(i) $s_{i+1j} s_{i+1} = s_i s_{i+1j}, j \leqslant i \leqslant l-2,$

(ii) $s_{lj} s_{lj} = s_{l-1} s_{lj} s_{lj+1}, j < l.$

Theorem. Relations (i),(ii) together with the initial relations of B_l form the reduced Gröbner-Shirshov basis of B_l. Corresponding reduced set consists of words of the form:

$$s_{1i_1} \cdots s_{l-1 i_{l-1}} s_{lj_1} \cdots s_{lj_k},$$

where $i_1 \leqslant 2, ..., i_{l-1} \leqslant l, 1 \leqslant j_1 < \cdots < j_k \leqslant l$ and $k \geqslant 0$.
Corollary 1. $|B_l| = l!2^l$.
Let $\{1, -1\}^l = (\epsilon_1, ..., \epsilon_l), \epsilon_i = \pm$ be the direct product of the 2-cyclic groups, $\mathrm{Sym}(l) \rtimes \{1, -1\}^l$ be the semidirect product such that $\mathrm{Sym}(l)$ acts on the second group by permutations.
Corollary 2. The Coxeter group B_l is isomorphic to $\mathrm{Sym}(l) \rtimes \{1, -1\}^l$ with the isomorphism $s_i \mapsto (i, i+1), i \leqslant l-1, s_l \mapsto (1, ..., 1, -1)$.

Finite Coxeter groups of type D_l

The Coxter group D_l is generated by s_i, $1 \leqslant i \leqslant l$ with the relations of A_{l-1} together with the relations
$s_l^2 = 1$, $s_l s_{l-1} = s_{l-1} s_l$, $s_l s_{l-2} s_l = s_{l-2} s_l s_{l-2}$.
Let us define s_{ij}, $1 \leqslant j \leqslant i+1 \leqslant l$, as before.
Put $s_{lj} = s_l s_{l-2} \dots s_j, j \leqslant l-2$; $s_{lj-1} = s_l$, $s_{ll} = 1$.

The following relations are valid in D_l:
(i) $s_{lj} s_{l-1j} = s_{l-1} s_{lj} s_{l-1j+1}$, $j \leqslant l-2$,
(ii)$s_{lj} s_{l-1} s_l = s_{l-2} s_{lj} s_{l-1}$, $j \leqslant l-2$,
(iii) $s_{lj} s_{l-1k} s_{lk} = s_{l-2} s_{lj} s_{l-1k} s_{lk+1}$, $j < k \leqslant l-2$.

Theorem. Relations (i), (ii), (iii) together with the Gröbner-Shirshov bases of A_{l-1}, and $A_{l-1}(s_1, \ldots, s_{l-2}, s_l)$ form the Gröbner-Shirshov basis of D_l. Corresponding reduced set consists of words of the form :

$$s_{1i_1} \cdots s_{l-1 i_{l-1}} s_{l j_1} s_{l-1 j_2} s_{l j_3} s_{l-1 j_4} \cdots ,$$

where $i_1 \leqslant 2, \ldots, i_{l-1} \leqslant l, 1 \leqslant j_1 < \cdots < j_k \leqslant l-1$ and $k \geqslant 0$.

Corollary. Number of that elements is $l!2^{l-1}$, and D_l is isomorphic to $Sym(l) \rtimes \{1, -1\}^{l-1}$

Remark. GS bases for exceptional Coxeter groups G_2, F_4, E_6, E_7 were found my students Denis Lee and for Coxeter group E_8 by Oleg Svechkarenko (the last result by PC).

Serre relations and GSB for simple Lie algebras

Serre relations of classical simple Lie algebras of types A_n, B_n, C_n, D_n over a field of characteristic $\neq 2, 3$ (just not 2 for A_n, D_n) in Weil generators $H = \{h_i, 1 \leqslant i \leqslant n\}$, $X = \{x_i, 1 \leqslant i \leqslant n\}$, $Y = \{y_i, 1 \leqslant i \leqslant n\}$ $(h > x > y)$ and corresponding Cartan matrices $C = ||\alpha_{ij}||$:

I. $h_i h_j = 0$, $i > j$

II. $x_i y_i - h_i = 0$,

III. $x_i y_j = 0$, $i \neq j$

IV. $h_i x_j - \alpha_{j,i} x_j = 0$

IV'. $h_i y_j + \alpha_{j,i} y_j = 0$

V. $(adx_i)^{1-\alpha_{j,i}}(x_j) = 0$, $i \neq j$

V'. $(ady_i)^{1-\alpha_{j,i}}(y_j) = 0$, $i \neq j$.

Serre relations and GSB for simple Lie algebras

Theorem (L.A.B., A.A.Klein, 1996a) For any field k of characteristic $\neq 2$, a Gröbner-Shirshov basis of A_n is given by the initial relations
I- V' and

VI. $[x_{i+k} \cdots x_{i-1}]x_{i+k-1}$, VI'. $[y_{i+k} \cdots y_{i-1}]y_{i+k-1}$, $i \geqslant 2, k \geqslant 0$,

VII. $[x_{i+k} \cdots x_i][x_{i+k} \cdots x_{i-1}]$,

VII'. $[y_{i+k} \cdots y_i][y_{i+k} \cdots y_{i-1}]$,$i \geqslant 2, k \geqslant 0$,

where $[f_1 \cdots f_n]$ is the right-normed form $f_1(\ldots(f_{n-2}(f_{n-1}f_n))\ldots)$.

Serre relations and GSB for simple Lie algebras

Corollary 1. The set of irreducible LS words of A_n is as follows:

$$\{h_1, ..., h_n\} \cup \{[x_{i+k} \cdots x_i], [y_{i+k} \cdots y_i] \mid n \geqslant i + k \geqslant i \geqslant 1\}.$$

Corollary 2. $\dim(A_n) = n^2 + 2n$.

Corollary 3 (Serre theorem). Lie algebra A_n is isomorphic to the Lie algebra sl_{n+1} of the trace zero $(n+1)$-matrices over k.

The same kind of theorems are proved for simple Lie algebras of type $B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$ (L.A.B., A.A.Klein,1996b, 1998, 1999)

Plactic monoid

Let $A = \{1, 2, \ldots, n\}$ with $1 < 2 < \cdots < n$. Then we call

$$Pl(A) := sgp\langle A|\Omega\rangle = A^*/\equiv$$

a plactic monoid on the alphabet set A, where A^* is the free monoid generated by A, $\equiv$ is the congruence of A^* generated by the Knuth relations Ω and Ω consists of

$$ikj = kij \ (i \leqslant j < k),$$
$$jki = jik \ (i < j \leqslant k).$$

Plactic monoid

Let $R \in A^*$ be a row. Then we denote $R = (r_1, r_2, \ldots, r_n)$, where r_i is the number of letter i ($i = 1, 2, \ldots, n$), for example,
$R = 1135556 = (2, 0, 1, 0, 3, 1, 0, \ldots, 0)$.

Let $U = \{R \in A^* \mid R \text{ is a row}\}$.

We order the set U^* as follows. Let $R = (r_1, r_2, \ldots, r_n) \in U$. Then $|R| = r_1 + \cdots + r_n$. We first order U: for any $R, S \in U$, $R < S$ if and only if $|R| < |S|$ or $|R| = |S|$ and there exists a t ($0 \leqslant t < n$) such that $r_i = s_i$, $i = 1, \ldots, t$ and $r_{t+1} > s_{t+1}$. Clearly, this is a well order on U. Then, we order U^* by the deg-lex order.

4	5	5	6			
2	2	3	3	5	7	
1	1	1	2	4	4	4

Figure: Young tableau

Lemma (Richardson - Shensted formula).There is the following formula in a plactic monoid $\mathcal{L}(A)$ for any row R and a letter $x \in A$:

$R \cdot x = Rx$, if Rx is a row; or $y \cdot R'$, otherwise, where y is the leftmost letter in R and is strictly larger than x, and $R' = R\mid_{y \mapsto x}$ a row, $y \cdot R'$ is a Young tableau.

Corollary. There is the following formula in the plactic monoid $\mathcal{L}(A)$ for any two rows W, Z:

$$W \cdot Z = X \cdot Y,$$

where X, Y are rows (Y may be empty) and XY the Young tableau.

Plactic monoid

Theorem (explicit formula for relations $W \cdot Z = X \cdot Y$ above).
Let $W = (w_1, w_2, \dots, w_n), Z = (z_1, z_2, \dots, z_n)$. Then
$X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n)$, $x_1 = 0$ and

$$x_p = min(Z_{p-1} - X_{p-1}, w_p), n \geqslant p \geqslant 2, \quad y_q = w_q + z_q - x_q, \;\; n \geqslant q \geqslant 1,$$

where $Z_k = z_1 + \cdots + z_k$, $X_k = x_1 + \cdots + x_k$.

Theorem. The GS basis of a plactic monoid in row generators relative the above order of rows consists of above relations $WZ = XY$, where XY is a Young tableau in rows.

Corollary (the book [46], Chapter 5) The set of Young tableaux on A is a set of normal forms of elements of the plactic monoid $\mathcal{L}(A)$.

Plactic monoid

Let $A = \{1, 2, \ldots, n\}$. Let $C \in A^*$ be a column. Let c_i be the number of letter i in C. Then $c_i \in \{0, 1\}$, $i = 1, 2, \ldots, n$.

We denote $C = (c_1; c_2; \ldots; c_n)$. For example, $C = 6421 = (1; 1; 0; 1; 0; 1; 0; \ldots; 0)$ and, say $n = 5$, then the Young tableau in column generations: $421 \cdot 521 \cdot 531$ will be denoted by $(1; 1; 0; 1; 0) \cdot (1; 1; 0; 0; 1) \cdot (1; 0; 1; 0; 1)$.

Let $V = \{C \mid C \text{ is a column in } A^*\}$.

Let $R = (r_1; r_2; \ldots; r_n) \in V$ and $wt(R) = (|R|, r_1, \ldots, r_n)$. We order V: for any $R, S \in V$, $R < S$ if and only if $wt(R) > wt(S)$ lexicograghically. Then, we order V^* by the deg-lex order.

Plactic monoid

Theorem. There is the following formula in the plactic monoid $\mathcal{L}(A)$ for any two columns W, Z:

$$W \cdot Z = X \cdot Y,$$

where X, Y are columns (Y may be empty) and XY the Young tableau. Explicitly, we have $W = (w_1, w_2, ..., w_n), Z = (z_1, z_2, ..., z_n)$, $X = (x_1, x_2, ..., x_n), Y = (y_1, y_2, ..., y_n)$, $y_p = min(W_p - Y_{p-1}, z_p), y_0 = 0$, $n \geqslant p \geqslant 2$, $x_q = w_q + z_q - y_q$, $n \geqslant q \geqslant 1$,
$Z_k = z_1 + ... + z_k, Y_k = y_1 + ... + y_k$.

Theorem. A finite GS basis of a plactic monoid in column generators relative the above order consists of relations $WZ = XY$ above, where XY is a Young tableau in columns.

Corollary (the book [46], Chapter 5) The set of Young tableaux on A is a set of normal forms of elements of the plactic monoid $\mathcal{L}(A)$.

Let

$$T_A = \{u = u^Y u^X | u^Y \in [Y],\ u^X \in ALSW(X)\}$$

and

$$T_N = \{[u] = u^Y [u^X] | u^Y \in [Y],\ [u^X] \in NLSW(X)\}.$$

(Composition-Diamond Lemma for $\mathbf{k}[Y] \otimes Lie(X)$) Let $S \subset \mathbf{k}[Y] \otimes Lie(X)$ be nonempty set of monic polynomials and $Id(S)$ be the $\mathbf{k}[Y]$-ideal of $\mathbf{k}[Y] \otimes Lie(X)$ generated by S. Then the following statements are equivalent.

(i) S is a Gröbner-Shirshov basis in $\mathbf{k}[Y] \otimes Lie(X)$.

(ii) $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b \in T_A$ for some $s \in S$ and $a, b \in [Y]X^*$.

(iii) $Irr(S) = \{[u] \mid [u] \in T_N,\ u \neq a\bar{s}b,$ for any $s \in S,\ a, b \in [Y]X^*\}$ is a $\mathbf{k}$-basis for $(\mathbf{k}[Y] \otimes Lie(X))/Id(S)$.

Non-special Lie algebra over commutative algebra

Let the field $\mathbf{k} = GF(2)$ and $K = \mathbf{k}[Y|R]$, where

$Y = \{y_i, i = 0, 1, 2, 3\},\ R = \{y_0 y_i = y_i\ (i = 0, 1, 2, 3),\ y_i y_j = 0\ (i, j \neq 0)\}.$

(A.I. Shirshov, 1953)Let $\mathcal{L}_{\mathrm{Sh}} = Lie_K(X|S_1, S_2)$, where $\mathbf{k} = GF(2)$ and $K = \mathbf{k}[Y|R]$,
$Y = \{y_i, i = 0, 1, 2, 3\},\ R = \{y_0 y_i = y_i\ (i = 0, 1, 2, 3),\ y_i y_j = 0\ (i, j \neq 0)\}$,
$X = \{x_i, 1 \leqslant i \leqslant 13\}$, S_1 consists of the following relations

$$\begin{aligned}
&[x_2, x_1] = x_{11},\ [x_3, x_1] = x_{13},\ [x_3, x_2] = x_{12},\\
&[x_5, x_3] = [x_6, x_2] = [x_8, x_1] = x_{10},\\
&[x_i, x_j] = 0 \quad (\text{for any other } i > j),
\end{aligned}$$

Non-special Lie algebra over commutative algebra

and S_2 consists of the following relations

$$
\begin{aligned}
& y_0x_i = x_i \ (i = 1, 2, \dots, 13), \\
& x_4 = y_1x_1, \ x_5 = y_2x_1, \ x_5 = y_1x_2, \ x_6 = y_3x_1, \ x_6 = y_1x_3, \\
& x_7 = y_2x_2, \ x_8 = y_3x_2, \ x_8 = y_2x_3, \ x_9 = y_3x_3, \\
& y_3x_{11} = x_{10}, \ y_1x_{12} = x_{10}, \ y_2x_{13} = x_{10}, \\
& y_1x_k = 0 \ (k = 4, 5, \dots, 11, 13), \\
& y_2x_t = 0 \ (t = 4, 5, \dots, 12), \\
& y_3x_l = 0 \ (l = 4, 5, \dots, 10, 12, 13).
\end{aligned}
$$

Theorem (L.A.Bokut,Yuqun Chen, Yongshan Chen, 2012). A GS basis of L_{Sh} is $S = S_1 \cup S_2 \cup RX \cup \{y_1x_2 = y_2x_1, \ y_1x_3 = y_3x_1, \ y_2x_3 = y_3x_2\}$.

Non-special Lie algebra over commutative algebra

Corrolary (A.I.Shirshov) The algebra $\mathcal{L}_{\mathsf{Sh}}$ is not special.

Since $x_{10} \in Irr(S)$ and $Irr(S)$ is a $\mathbf{k}$-basis of $\mathcal{L}$ by Theorem 22, $x_{10} \neq 0$ in $\mathcal{L}_{\mathsf{Sh}}$.
On the other hand, the universal enveloping algebra of $\mathcal{L}_{\mathsf{Sh}}$ has a presentation:

$$U_K(\mathcal{L}_{\mathsf{Sh}}) = K\langle X | S_1^{(-)}, S_2\rangle \cong \mathbf{k}[Y]\langle X | S_1^{(-)}, S_2, RX\rangle,$$

where $S_1^{(-)}$ is just S_1 but substitute all $[uv]$ by $uv - vu$.
But the Gröbner-Shirshov complement of $S_1^{(-)} \cup S_2 \cup RX$ in $\mathbf{k}[Y]\langle X\rangle$, is

$$S^C = S_1^{(-)} \cup S_2 \cup RX \cup \{y_1x_2 = y_2x_1,\ y_1x_3 = y_3x_1,\ y_2x_3 = y_3x_2,\ x_{10} = 0\}.$$

Thus, $\mathcal{L}_{\mathsf{Sh}}$ is not special.

Non-special Lie algebra over commutative algebra

(P.Cartier) $\mathcal{L}_{\mathrm{Ca}} = Lie_K(X|S)$, where $\mathbf{k} = GF(2)$,
$K = \mathbf{k}[y_1, y_2, y_3 | y_i^2 = 0,\ i = 1, 2, 3]$, $X = \{x_{ij}, 1 \leqslant i \leqslant j \leqslant 3\}$ and

$$S = \{[x_{ii}, x_{jj}] = x_{ji}\ (i > j), [x_{ij}, x_{kl}] = 0\ (\text{others}),\ y_3 x_{33} = y_2 x_{22} + y_1 x_{11}\}.$$

Theorem $S' = S \cup \{y_i^2 x_{kl} = 0\ (\forall i, k, l)\} \cup S_1$ is a Gröbner-Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$, where S_1 consists of the following relations

$$y_3 x_{23} = y_1 x_{12},\ y_3 x_{13} = y_2 x_{12},\ y_2 x_{23} = y_1 x_{13},\ y_3 y_2 x_{22} = y_3 y_1 x_{11},$$
$$y_3 y_1 x_{12} = 0,\ y_3 y_2 x_{12} = 0,\ y_3 y_2 y_1 x_{11} = 0,\ y_2 y_1 x_{13} = 0.$$

Corollary(P.Cartier, 1958) $\mathcal{L}_{\mathrm{Ca}}$ is not special.

The universal enveloping algebra of $\mathcal{L}_{\mathrm{Ca}}$ has a presentation:

$$PBW_K(\mathcal{L}_{\mathrm{Ca}}) = K\langle X|S^{(-)}\rangle \cong \mathbf{k}[Y]\langle X|S^{(-)}, y_i^2 x_{kl} = 0\ (\forall i,k,l)\rangle.$$

In $PBW_K(\mathcal{L}_{\mathrm{Ca}})$, we have

$$0 = y_3^2 x_{33}^2 = (y_2x_{22} + y_1x_{11})^2 = y_2^2x_{22}^2 + y_1^2x_{11}^2 + y_2y_1[x_{22}, x_{11}] = y_2y_1x_{12}.$$

On the other hand, since $y_2y_1x_{12} \in Irr(S')$, $y_2y_1x_{12} \neq 0$ in $\mathcal{L}_{\mathrm{Ca}}$. Thus, $\mathcal{L}_{\mathrm{Ca}}$ is not special.

Non-special Lie algebra over commutative algebra

(P.M. Cohn,1963) Let

$$\mathcal{L}_{\mathrm{Co},p} = Lie_K(x_1, x_2, x_3 \mid y_3x_3 = y_2x_2 + y_1x_1),$$

where $K = \mathbf{k}[y_1, y_2, y_3|y_i^p = 0, i = 1, 2, 3]$ is an algebra of truncated polynomials over a field k of characteristic $p > 0$.
First, we consider $p = 2$ and prove the element
$\Lambda_2 = [y_2x_2, y_1x_1] = y_2y_1[x_2x_1] \neq 0$ in $\mathcal{L}_{\mathrm{Co},2}$.
Let S_{X^2} be the set of all the elements of $\mathrm{GSB}(S)$ whose X-degrees do not exceed 2. Then by Shirshov's algorithm we have that S_{X^2} consists of the following relations

$$y_3x_3 = y_2x_2 + y_1x_1,\ y_i^2x_j = 0,\ y_3y_2x_2 = y_3y_1x_1,\ y_3y_2y_1x_1 = 0,$$
$$y_2[x_3x_2] = y_1[x_3x_1],\ y_3y_1[x_2x_1] = 0,\ y_2y_1[x_3x_1] = 0.$$

Thus, Λ_2 is in the $\mathbf{k}$-basis $Irr(\mathrm{GSB}(S))$ of $\mathcal{L}_{\mathrm{Co},2}$ and $\mathcal{L}_{\mathrm{Co},2}$ is not special.

Second, we consider $p = 3$ and prove the element
$\Lambda_3 = y_2^2 y_1[x_2x_2x_1] + y_2y_1^2[x_2x_1x_1] \neq 0$ in $\mathcal{L}_{\mathrm{Co},3}$.
Let S_{X^3} be the set of all the elements of GSB(S) whose X-degrees do not exceed 3. Then again by Shirshov's algorithm, S_{X^3} consists of the following relations

$$
\begin{aligned}
& y_3x_3 = y_2x_2 + y_1x_1,\ y_i^3x_j = 0,\ y_3^2y_2x_2 = y_3^2y_1x_1,\ y_3^2y_2^2y_1x_1 = 0,\\
& y_2[x_3x_2] = -y_1[x_3x_1],\ y_3^2y_1[x_2x_1] = 0,\ y_2^2y_1[x_3x_1] = 0,\\
& y_3y_2^2[x_2x_2x_1] = y_3y_2y_1[x_2x_1x_1],\ y_3y_2^2y_1[x_2x_1x_1] = 0,\\
& y_3y_2y_1[x_2x_2x_1] = y_3y_1^2[x_2x_1x_1].
\end{aligned}
$$

Thus, $y_2^2y_1[x_2x_2x_1], y_2y_1^2[x_2x_1x_1] \in Irr(\mathrm{GSB}(S))$, which implies $\Lambda_3 \neq 0$ in $\mathcal{L}_{\mathrm{Co},3}$.

Therefore,
(P.M. Cohn) Lie algebras $\mathcal{L}_{\mathrm{Co},2}$ and $\mathcal{L}_{\mathrm{Co},3}$ are not special.

Let X be an oriented multi-graph, $C(X)$ the free category generated by X and $kC(X)$ the free category (partial) algebra.

(CD-lemma for categories)TFAE:

(i) S is a Gröbner-Shirshov basis in $kC(X)$.

(ii) $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in C(X)$.

(iii) $Irr(S) = \{u \in C(X) | u \neq a\bar{s}b \ a, b \in C(X), \ s \in S\}$ is a linear basis of the category (partial) algebra $kC(X)/Id(S)$ which is denoted by $kC(X|S)$.

GSB for simplicial category

For each non-negative integer p, let $[p]$ denote the set $\{0, 1, 2, \ldots, p\}$ of integers in their usual ordering. A (weakly) monotonic map $\mu : [q] \to [p]$ is a function on $[q]$ to $[p]$ such that $i \leqslant j$ implies $\mu(i) \leqslant \mu(j)$. The objects $[p]$ with morphisms all weakly monotonic maps μ constitute a category $\mathscr{L}$ called simplicial category. It is convenient to use two special families of monotonic maps

$$\varepsilon_q^i : [q-1] \to [q], \quad \eta_q^i : [q+1] \to [q]$$

defined for $i = 0, 1, ...q$ (and for $q > 0$ in the case of ε^i) by

$$\varepsilon_q^i(j) = \begin{cases} j, & \text{if } \ i > j, \\ j+1, & \text{if } \ i \leqslant j, \end{cases}$$

$$\eta_q^i(j) = \begin{cases} j, & \text{if } \ i \geqslant j, \\ j-1, & \text{if } \ j > i. \end{cases}$$

It is easy to check the following relations in $\mathscr{L}$:

$$\begin{aligned}
f_{q+1,q}: &\quad \varepsilon^i_{q+1}\varepsilon^{j-1}_q = \varepsilon^j_{q+1}\varepsilon^i_q, \quad j > i, \\
g_{q,q+1}: &\quad \eta^j_q\eta^i_{q+1} = \eta^i_q\eta^{j+1}_{q+1}, \quad j \geqslant i, \\
h_{q-1,q}: &\quad \eta^j_{q-1}\varepsilon^i_q = \begin{cases} \varepsilon^i_{q-1}\eta^{j-1}_{q-2}, & j > i, \\ 1_{q-1}, & i = j, \quad i = j+1, \\ \varepsilon^{i-1}_{q-1}\eta^j_{q-2}, & i > j+1. \end{cases}
\end{aligned}$$

where
$E = \{\varepsilon^i_p : [p-1] \to [p],\ \eta^j_q : [q+1] \to [q] \mid p > 0, 0 \leqslant i \leqslant p, 0 \leqslant j \leqslant q\}.$

GSB for simplicial category

Theorem (L.A.B., Yuqun Chen, Yu Li, 2012). Let $C(E|S)$ be an absract category genereted by
$E = \{\varepsilon_p^i : [p-1] \to [p],\ \eta_q^j : [q+1] \to [q] \mid p > 0, 0 \leqslant i \leqslant p, 0 \leqslant j \leqslant q\}$
with definibg relations $S = \{f_{p+1,p}, g_{q,q+1}, h_{t-1,t}\}$ above. Then S is a GS basis of $C(E|S)$ and any element $\mu : [q] \to [p]$ of this category has a unique presentation in the form

$$\varepsilon_p^{i_1} \dots \varepsilon_{p-m+1}^{i_m} \eta_{q-n}^{j_1} \dots \eta_{q-1}^{j_n},$$

where $p \geqslant i_1 > \dots > i_m \geqslant 0, \quad 0 \leqslant j_1 < \dots < j_n < q,$ and $q - n + m = p.$

Corollary. The simplitial category $\mathscr{L}$ is isomorphic the abstract category $C(E|S)$ above.

GSB for cyclic category

Let $C(F|S)$ be the abstact category generated by F with defining relations S,
where $F = \{\varepsilon_p^i : [p-1] \to [p],\ \eta_q^j : [q+1] \to [q],\ t_q : [q] \to [q] \mid p > 0, 0 \leqslant i \leqslant p, 0 \leqslant j \leqslant q\}$ and S consists of the following relations

$$
\begin{aligned}
f_{q+1,q} &: \quad \varepsilon_{q+1}^i \varepsilon_q^{j-1} = \varepsilon_{q+1}^j \varepsilon_q^i, \quad j > i, \\
g_{q,q+1} &: \quad \eta_q^j \eta_{q+1}^i = \eta_q^i \eta_{q+1}^{j+1}, \quad j \geqslant i, \\
h_{q-1,q} &: \quad \eta_{q-1}^j \varepsilon_q^i = \begin{cases} \varepsilon_{q-1}^i \eta_{q-2}^{j-1}, & j > i, \\ 1_{q-1}, & i = j, \quad i = j+1, \\ \varepsilon_{q-1}^{i-1} \eta_{q-2}^j, & i > j+1, \end{cases} \\
\rho_1 &: \quad t_q \varepsilon_q^i = \varepsilon_q^{i-1} t_{q-1}, \quad i = 1, \dots, q, \\
\rho_2 &: \quad t_q \eta_q^i = \eta_q^{i-1} t_{q+1}, \quad i = 1, \dots, q, \\
\rho_3 &: \quad t_q^{q+1} = 1_q.
\end{aligned}
$$

Theorem . The set S together with $\rho_4: \quad t_q\varepsilon_q^0 = \varepsilon_q^q,\ \rho_5: \quad t_q\eta_q^0 = \eta_q^q t_{q+1}^2$ is a GS basis of $C(F|S)$.

Corrolary 1. Normal form of an elements $\mu : [q] \to [p]$ in the category $C(F|S)$ is

$$\varepsilon_p^{i_1} \ldots \varepsilon_{p-m+1}^{i_m} \eta_{q-n}^{j_1} \ldots \eta_{q-1}^{j_n} t_q^k,$$

where

$p \geqslant i_1 > \ldots > i_m \geqslant 0,\ \ 0 \leqslant j_1 < \ldots < j_n < q,\ 0 \leqslant k \leqslant q$ and $q - n + m = p$.

Corollary 2 (A. Connes, 1983). The cyclic category Λ is isomorphic the the category $C(F|S)$.

GSB for cyclic category

Remark. Multiple descriptions of the cycle category Λ are possible, but a convenient starting point is to consider first a category L whose objects are natural numbers $n \geqslant 0$, and where the hom-set $L(m, n)$ consists of increasing functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the "spiraling property", that $f(i + m + 1) = f(i) + n + 1$, with composition given by functional composition. Then, define Λ to be a quotient category of L having the same objects, with $\Lambda(m, n) = L(m, n)/\sim$, where $\sim$ is the equivalence relation for which $f \sim g$ means $f - g$ is a constant multiple of $n + 1$. Let $q : L \to \Lambda$ be the quotient.

Basis of free semiring $Rig\langle X\rangle$:

$w = u_1 \circ u_2 \circ \cdots \circ u_n$, where $u_i \in X^*, u_1 \leqslant u_2 \leqslant \ldots \leqslant u_n,\ n \geqslant 0$ and
$w = \theta$ if $n = 0$.

(Composition-Diamond lemma for semirings) TFAE

(1) S is a Gröbner-Shirshov basis in $kRig\langle X\rangle$.

(2) $f \in Id(S) \Rightarrow \overline{f} = a\overline{s}b \circ u$ for some $a, b \in X^*, u \in Rig\langle X\rangle$ and $s \in S$.

(2′) $f \in Id(S) \Rightarrow f = \alpha_1 a_1 s_1 b_1 \circ u_1 + \alpha_2 a_2 s_2 b_2 \circ u_2 + \ldots + \alpha_n a_n s_n b_n \circ u_n$,
where $a_1\overline{s_1}b_1 \circ u_1 > a_2\overline{s_2}b_2 \circ u_2 > \ldots > a_n\overline{s_n}b_n \circ u_n$,
$0 \neq \alpha_i \in k,\ a_i, b_i \in X^*,\ u_i \in Rig\langle X\rangle,\ s_i \in S$.

(3) $Irr(S) = \{w \in Rig\langle X\rangle \mid w \neq a\overline{s}b \circ u$ for any $a, b \in X^*, u \in Rig\langle X\rangle, s \in S\}$ is a k-basis of $kRig\langle X|S\rangle = kRig\langle X\rangle/Id(S)$.

CD lemma for semirings

In 1995, A. Blass finds a normal form of the semiring $N[x]/(x = 1 + x^2)$. Clearly, $N[x]/(x = 1 + x^2) = Rig[x|x = 1 \circ x^2]$.

A Gröbner-Shirshov basis of $Rig[x|x = 1 \circ x^2]$ consists of the following relations

1. $1 \circ x^2 = x$,
2. $x \circ x^4 = 1 \circ x^3$,
3. $x^5 = 1 \circ x^4$,
4. $1 \circ x^3 \circ x^n = x^n \ (3 \leqslant n \leqslant 4)$.

A normal form of the semiring $Rig[x|x = 1 \circ x^2]$ is the set

$\{(1^{\circ n} \circ x^{\circ m})x^t, 1^{\circ n} \circ x^3, 1^{\circ n} \circ (x^4)^{\circ m} | n, m \geqslant 0, 0 \leqslant t \leqslant 3\}$.

CD lemma for semirings

A Gröbner-Shirshov basis of $Rig[x|x = 1 \circ x \circ x^2]$ consists of the following relations

1. $x^4 = 1 \circ 1 \circ x^2$,
2. $x \circ x^3 = 1 \circ x^2$,
3. $1 \circ x^2 \circ x^n = x^n \quad (1 \leqslant n \leqslant 3)$.

A normal form of the semiring $Rig[x|x = 1 \circ x \circ x^2]$ is the set

$$\{1^{\circ(m+1)} \circ x^2,\ 1^{\circ m} \circ x^{\circ n},\ 1^{\circ m} \circ (x^3)^{\circ n},\ x^{\circ m} \circ (x^2)^{\circ n},\ (x^2)^{\circ m} \circ (x^3)^{\circ n} | n, m \geqslant 0\}.$$

The normal form is the same as that in: M. Fiore and T. Leinster (2004).

Metabelian Poisson algebra

We recall that a *Poisson algebra* is a vector space $\mathcal{A}$ over a field k endowed with two bilinear operations, a multiplication written $\cdot$ and a Poisson bracket written $(-,-)$, such that $(\mathcal{A},\cdot)$ is a commutative associative algebra, $(\mathcal{A},(-,-))$ is a Lie algebra, and $\mathcal{A}$ satisfies the *Leibniz identity*

$$(x, y\cdot z) = (x,y)\cdot z + y\cdot(x,z). \tag{1}$$

A Poisson algebra $\mathcal{A}$ is called *abelian* if $\mathcal{A}$ is just a vector space with trivial product, that is, every product of arbitrary two elements is 0, and,

a Poisson algebra $\mathcal{A}$ is called *metabelian* if $\mathcal{A}$ is an extension of one abelian Poisson algebra by another abelian Poisson algebra.

Metabelian Poisson algebra

Let X be a well-ordered set, and define the sets $Y_1, \dots, Y_5$ as
$Y_1 = \{[a_1, \dots, a_n]_{\mathrm{L}} \mid a_1, \dots, a_n \in X, n \geqslant 2, a_1 > a_2, a_2 \leqslant \dots \leqslant a_n\}$,
$Y_2 = \{a_1 \cdot \dots \cdot a_t \mid a_1, \dots, a_t \in X, a_1 \leqslant \dots \leqslant a_t, 1 \leqslant t \leqslant 3\}$,
$Y_3 = \{(a_1, a_2) \cdot a_3 \mid a_1, a_2, a_3 \in X, a_1 > a_2\}$,
$Y_4 = \{((a_1, a_2), a_3) \cdot a_4 \mid a_1, \dots, a_4 \in X, a_2 < a_1 \leqslant a_4, a_2 \leqslant a_3 < a_4\}$,
$Y_5 = \{[a_1, \dots, a_n]_{\mathrm{L}} \cdot a_{n+1} \mid a_1, \dots, a_{n+1} \in X, n \geqslant 3, a_1 > a_2, a_2 \leqslant \dots \leqslant a_{n+1}\}$.
Define

$$\mathcal{MPB}_k(X) := \begin{cases} Y_1 \cup Y_2 \cup Y_3 \cup Y_4, & \text{if } \operatorname{char}(k) \neq 2, \\ Y_1 \cup Y_2 \cup Y_3 \cup Y_5, & \text{if } \operatorname{char}(k) = 2. \end{cases}$$

Then $\mathcal{MPB}_k(X)$ is a linear generating set for $\mathcal{MP}(X)$.

Let X be a finite set. Then $\mathcal{MP}(X \mid S)$ has a finite Gröbner–Shirshov basis. In particular, the word problem for finitely presented metabelian Poisson algebra $\mathcal{MP}(X \mid S)$ is solvable.

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Thank you!