Combinatorial Games on Graphs, Coxeter-Dynkin diagrams, and the Geometry of Root Systems

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ADE graphs

\begin{align*}
A_n & \quad \begin{array}{c}
\bullet \\
1 & 2 & \cdots & n
\end{array} \\
D_n & \quad \begin{array}{c}
\bullet \\
1 & 2 & \cdots & n
\end{array} \\
E_6 & \quad \begin{array}{c}
\bullet \\
1 & 2 & 3 & 4 & 5
\end{array} \\
E_7 & \quad \begin{array}{c}
\bullet \\
1 & 2 & 3 & 4 & 5 & 6
\end{array} \\
E_8 & \quad \begin{array}{c}
\bullet \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
\end{align*}

Sp(X) < 2

\begin{align*}
A_n^\sim & \quad \begin{array}{c}
\bullet \\
1 & 2 & \cdots & n
\end{array} \\
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\end{array}
\end{align*}

Sp(X) = 2

\begin{align*}
Sp(X) > 2
\end{align*}

\text{ADE} \sim \text{graphs}
Disclosure: I do not believe in:

a) “irrational numbers”
b) “infinite processes that can be completed”
c) “infinite sets”
d) “axioms” as a basis for mathematics

Since “eigenvalues” are problematic, we need alternate ways to describe the ADE / ADE~ / other division.
The $SP(X)=2$ Perron Frobenius vectors on $ADE\sim$ graphs
ADE graphs and Platonic solids

A\_n: Cyclic \{1,1,n\}
D\_n: Dihedral \{2,2,n-2\}
E\_6: Tetrahedral \{2,3,3\}
E\_7: Octahedral / Cube \{2,3,4\}
E\_8: Icosahedron / Dodecahedron \{2,3,5\}

\{p,q,r\} : p faces around an edge, q edges around a vertex, r vertices around a face (or dual)

\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1
\]
W. Killing (1888) classified simple Lie algebras

- $A_n$: $\text{sl}(n)$
- $B_n$: $\text{so}(2n-1)$
- $C_n$: $\text{sp}(2n)$
- $D_n$: $\text{so}(2n)$
- $E_6$: $\dim = 78$
- $E_7$: $\dim = 133$
- $E_8$: $\dim = 248$
- $F_4$: $\dim = 52$
- $G_2$: $\dim = 14$

Simple Lie algebras $\rightarrow$ Root systems $\rightarrow$ Dynkin diagrams
Lie groups, symmetric spaces, reflection groups

Simple Lie algebras $\rightarrow$ Lie groups $\rightarrow$ Symmetric spaces

Weyl groups (generalizations of $S_n$)

Coxeter groups (generated by reflections)
Two other occurrences of ADE graphs

Pierre Gabriel (1972): quivers of finite type and indecomposable representations

John McKay (1979): ADE graphs correspond to finite subgroups of SU(2) / unit quaternions

The binary polyhedral groups are:

- $A_n$: binary cyclic group of an $(n + 1)$-gon, order $2n$
- $D_n$: binary dihedral group of an $n$-gon, $\langle 2, 2, n \rangle$, order $4n$
- $E_6$: binary tetrahedral group, $\langle 2, 3, 3 \rangle$, order 24
- $E_7$: binary octahedral group, $\langle 2, 3, 4 \rangle$, order 48
- $E_8$: binary icosahedral group, $\langle 2, 3, 5 \rangle$, order 120

The 600-cell ($E_8$)

$1 + 2 + 3 + 4 + 5 + 6 + 4 + 2 + 3 = 120$
Many other occurrences of ADE graphs !!

Von Neumann algebras and II-1 factors

Conformal field theory and Wess-Zumino-Witten models (fusion rule algebras) String theory!

Catastrophe theory

Simple singularities of holomorphic functions (V I Arnold)

Combinatorics!!
The Mutation Game

\( X = \) simple, connected graph. A population on \( X \) is an integer valued function on the vertices.

\( P(X) = \) populations on \( X \)

The mutation \( s_y \) at the vertex \( y \): fixes all population values except that at \( y \), which gets replaced by its negative plus the sum of its neighbours.

\[
p = [ -1,2,4,3 ]
\]

\[
ps_y = [ -1,4,4,3 ]
\]
E6 mutation sequence

From a singleton population to the maximal population

\[ s_0 \]
\[ s_2 \]
\[ s_3 \]
\[ s_4 \]
\[ s_5 \]
A root population is a population obtained from a singleton population by applying any sequence of mutations. $R(X)$ denotes the root populations of the graph $X$.

$$R(D4) = R^+(D4) + R^-(D4)$$

Where a root is positive if all its entries are positive ($\geq 0$).
**ADE Graphs**

**Theorem:** \( R(X) \) is finite precisely when \( X \) is an ADE graph, i.e. in this following list:

These sets \( R(X) \) turn out to be the irreducible (simply laced) root systems studied by E. Dynkin.

These are sets of vectors in a Euclidean space satisfying symmetry wrt reflections in hyperplanes.
Proof:

1. If $X$ is ADE then $R(X)$ is finite (enumerate them!)

2. If $X$ is not ADE, then it contains an ADE~ subgraph

3. Show that if $X$ is ADE~ then $R(X)$ is unbounded: use the Perron Frobenius vector which is unchanged by mutations, for example for $E_6$:

\[ \delta_3 s(2, 4, 0, 1, 5, *, 3, 2, 4, 0, 3) = \delta_3 + \alpha_0 \]
An important conjecture

The Mutation Fact / Conjecture: For any simple connected graph X, R(X) is always the disjoint union of positive and negative roots.

This is known as a consequence of the theory of Coxeter groups, generated by reflections. (Personal communication with Bob Howlett). However we do not have a combinatorial proof/understanding.

A restatement: a root population can never have both strictly positive and strictly negative entries, for any graph X!!
Positive root populations of E6

$|R(E6)| = 36 + 36 = 72$
A2: The two-dimensional mutation representation of $W(A2) = S_3 = \langle s_1, s_2 \rangle$

\[
\begin{align*}
\begin{pmatrix}
-1 & 0 \\
1 & 0
\end{pmatrix} & \implies (a+b, b) \\
\begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix} & \implies (a, a-b) \\
\begin{pmatrix}
1 & 1 \\
0 & -1
\end{pmatrix} & \implies (1, 1)
\end{align*}
\]

Reflection relation:

$S_1^2 = s_2^2 = 1$

Braid relation:

$s_1 s_2 s_1 = s_2 s_1 s_2$
A3: The three-dimensional mutation representation of $W(A3) = S_4 = \langle s_1, s_2, s_3 \rangle$ and the Permutahedron

$\begin{align*}
(a, b, c) \\
(a, a-b+c, c) \\
(-b+c, a-b+c, c) \\
(c-b, -a+c, c) \\
(c-b, -a+c, -a) \\
(c-b, -b, -a) \\
(-c, -b, -a)
\end{align*}$

Longest word in $W$
The Tits Quadratic Form

Define a symmetric bilinear form on \( P(X) = \) populations on \( X \) via the symmetric matrix \( C = 2I - A \) [Cartan matrix] where \( A \) is the adjacency of the graph.

\[
Q(a, b, c, d) = 2a^2 + 2b^2 + 2c^2 + 2d^2 - ab - bc - bd
\]

**Theorem:** The mutations \( s_x \) are isometries with respect to the Tits quadratic form. So \( W = \langle s_x \rangle \) is a group of isometries.
A2 lattice, sphere $Q(v)=2$ and root system

$Q((a,b))=2a^2+2b^2-ab$
Rational Trigonometry

A symmetric bilinear form gives geometry! The algebraic approach forgets about distances and angles, and uses \textit{quadrance} and \textit{spread}!

\[ s((x_1, y_1), (x_2, y_2)) = \frac{(x_1y_2 - x_2y_1)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \]

Extends to general quadratic forms!
Old Babylonian Trigonometry

Plimpton 322 from 1800 B.C.E. is the world's first trigonometric table: using ratio-based trigonometry!

[Plimpton 322 is Babylonian exact sexagesimal trigonometry, Mansfield D., Wildberger N.J., 2017 Historia Mathematica]
A generalized (simply-laced) root system is a type of vector in an inner product space, each with the same quadrance, invariant under reflections in any associated perpendicular hyperplane, with reflections given by integral multiples of root vectors.

Theorem: $R(X)$ for any graph $X$ is a generalized simply-laced root system. It is finite precisely when $X$ is ADE.
Root hyperplanes, Weyl chambers and orbits

A Weyl group $W$ orbit for A2
Size of root populations and Weyl groups $W = \langle s_x \rangle$

| $|R(An)|$ | $n^2 + n$ | $|W(An)|$ | $(n + 1)!$ |
|----------|----------|----------|-----------|

| $|R(Dn)|$ | $2n^2 - 2n$ | $|W(Dn)|$ | $2^n \cdot n!$ |

| $|R(E6)|$ | $36 + 36 = 72$ | $|W(E6)|$ | $51,840$ |

| $|R(E7)|$ | $63 + 63 = 126$ | $|W(E7)|$ | $2,903,040$ |

| $|R(E8)|$ | $120 + 120 = 240$ | $|W(E8)|$ | $696,729,600$ |
**Theorem:** The Tits quadratic form is degenerate precisely when $X$ is an ADE~ graph.

Then spectral radius $(X) = 2$, and a Perron Frobenius vector has quadrance $Q(v) = 0$.

Remove the ~ node, and you get the maximum root population on the associated ADE graph.
Remarkable lattices

In the case of an **ADE graph**, $P(X)$ is a (Euclidean) geometric lattice since the Tits quadratic form is positive definite. But these lattices also have remarkable properties!

<table>
<thead>
<tr>
<th>Dim</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>&gt; 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best Lattice Packing</td>
<td>A1</td>
<td>A2</td>
<td>A3</td>
<td>D4</td>
<td>D5</td>
<td>E6</td>
<td>E7</td>
<td>E8</td>
<td>??</td>
</tr>
<tr>
<td>Largest Kissing Number</td>
<td>A1</td>
<td>A2</td>
<td>A3</td>
<td>D4</td>
<td>D5</td>
<td>E6</td>
<td>E7</td>
<td>E8</td>
<td>…</td>
</tr>
<tr>
<td>Number of sphere neighbours</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td>40</td>
<td>72</td>
<td>126</td>
<td>240</td>
<td>…</td>
</tr>
</tbody>
</table>
An **E7** Mutation Sequence

And an associated mutation frame... the swallow!
Another E7 Mutation Sequence
An E8 Mutation Sequence
The associated $E_8$ Mutation Frame

This is a pomset: a partially ordered multiset!
The swallow: the E7 X-heap

**Theorem 2.1** Let \( X \) be a simple graph for which there exists a maximal neighbourly X-heap \( F \) which is two-neighbourly. Then \( X \) is one of the graphs \( A_n, n \geq 1, \ D_n, \ n \geq 4, \ E_6 \) or \( E_7 \). There are exactly \( n \) such X-heaps for \( A_n \), three for \( D_n \), two for \( E_6 \) and one for \( E_7 \).

In “A Combinatorial Construction of simply-laced Lie algebras” (2003), I show how to construct ADE Lie algebras except E8 through minuscule representations via spaces of ideals on such heaps.

For E7 this gives the smallest 56 dim representation.

In another paper I give a similar realization of the 14 dim rep of G2.
Lie algebra representations and weights

15 dim representation of $\text{sl}(3)$ or $\text{su}(3)$
Mutations, Root systems and related heaps/lattices/pomsets on general graphs $X$:

A huge area of potential investigation!

Thanks for listening, and many thanks to the organizers of G2 R2!

Join the Algebraic Calculus One course: n.wildberger@unsw.edu.au