Combinatorial Games on Graphs, Coxeter-Dynkin diagrams, and the Geometry of Root Systems

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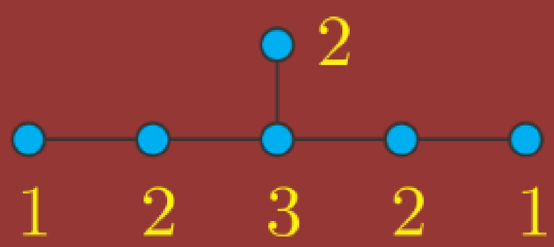
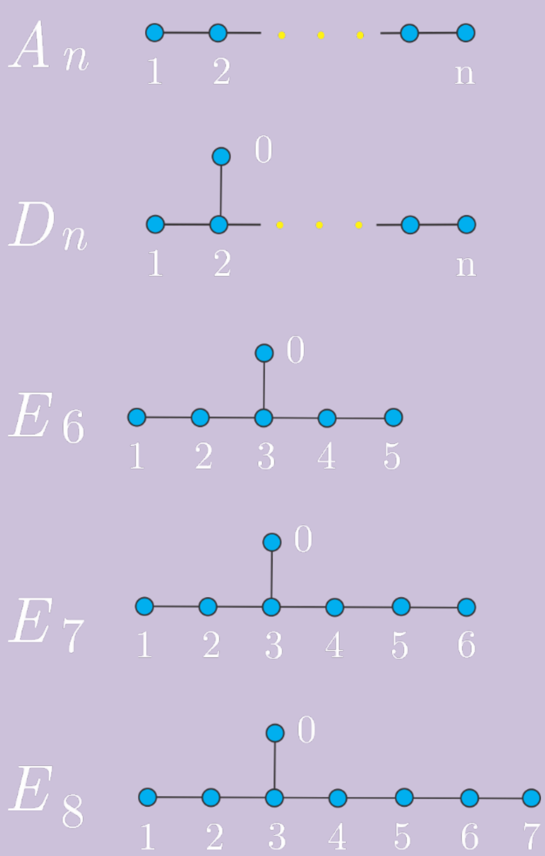


Graphs and Groups, Representations and Relations

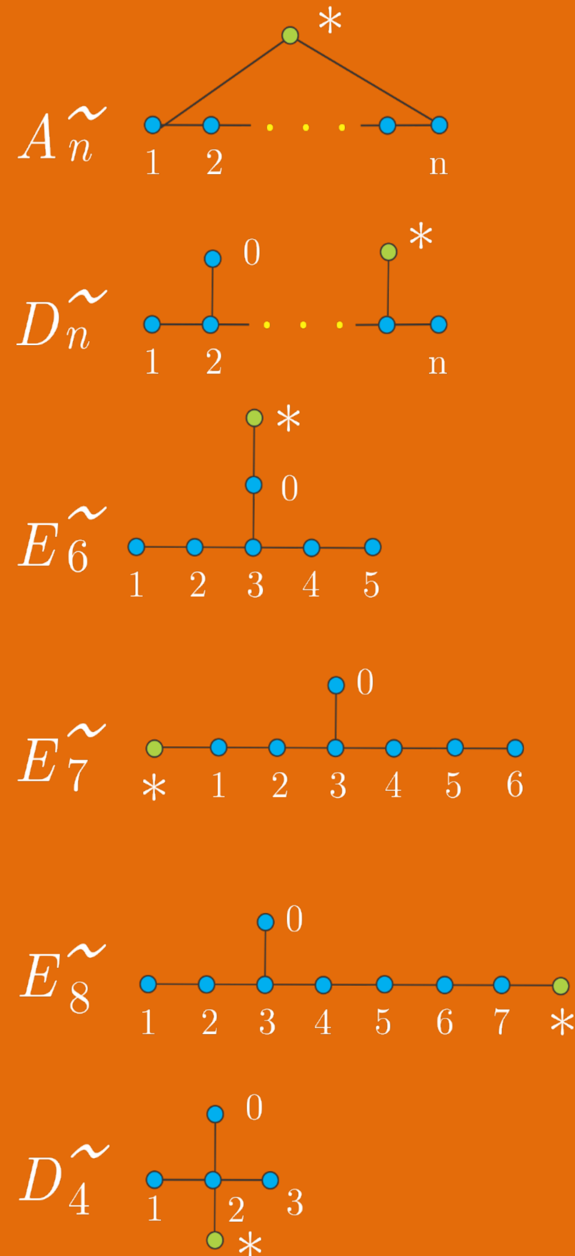
Novosibirsk August 6-19, 2018

ADE graphs



Sp(X) < 2

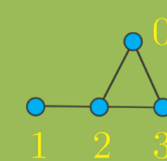
ADE ~ graphs



Sp(X) = 2

All other graphs

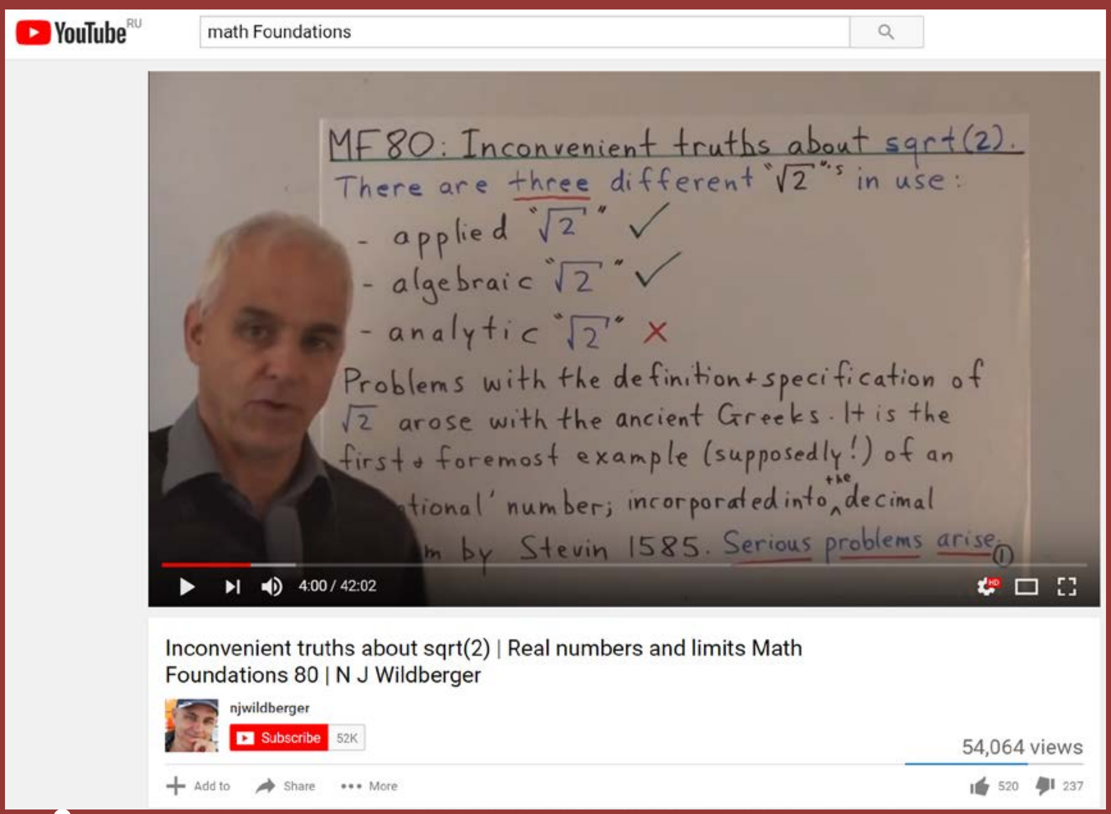
. . .



Sp(X) > 2

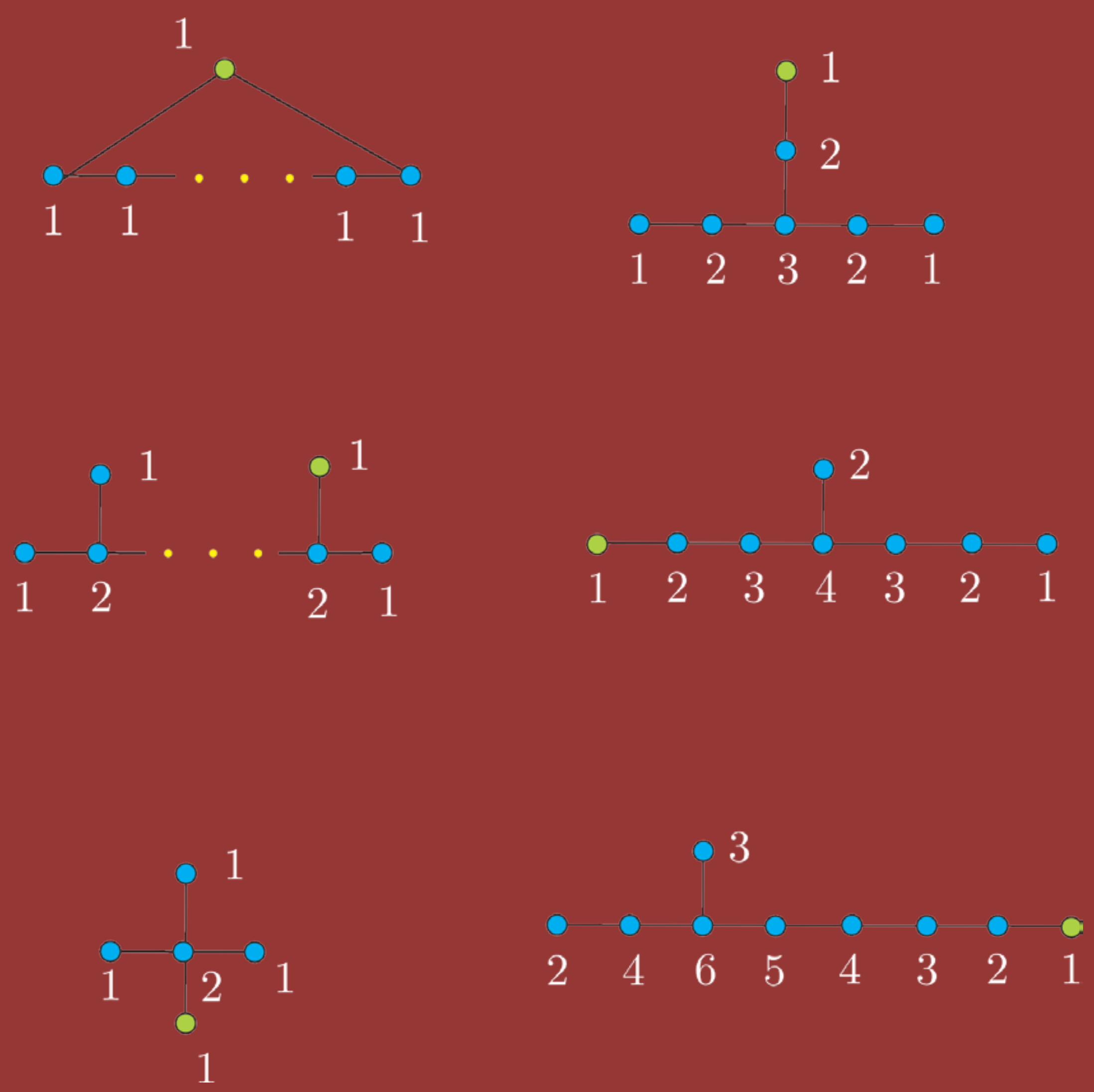
Disclosure: I do not believe in:

a) “irrational numbers”
b) “infinite processes that can be completed”
c) “infinite sets”
d) “axioms” as a basis for mathematics



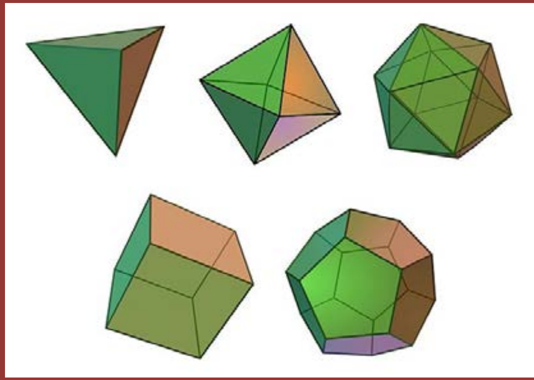
Since “eigenvalues” are problematic, we need alternate ways to describe the ADE / ADE~ / other division.

The SP(X)=2 Perron Frobenius vectors on ADE~ graphs

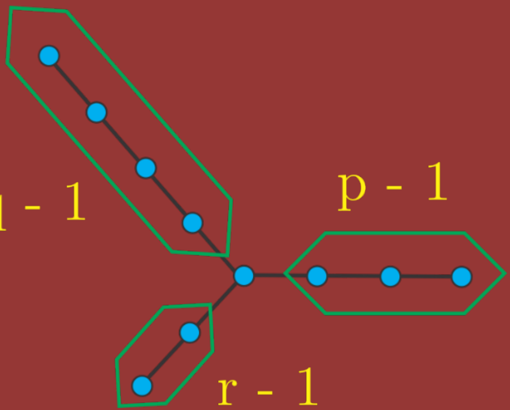


ADE graphs and Platonic solids

A_n: Cyclic {1,1,n}
D_n: Dihedral {2,2,n-2}
E_6: Tetrahedral {2,3,3}
E_7: Octahedral / Cube {2,3,4}
E_8: Icosahedron / Dodecahedron {2,3,5}



{p,q,r} : p faces around an edge, q edges around a vertex, r vertices around a face (or dual)



$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

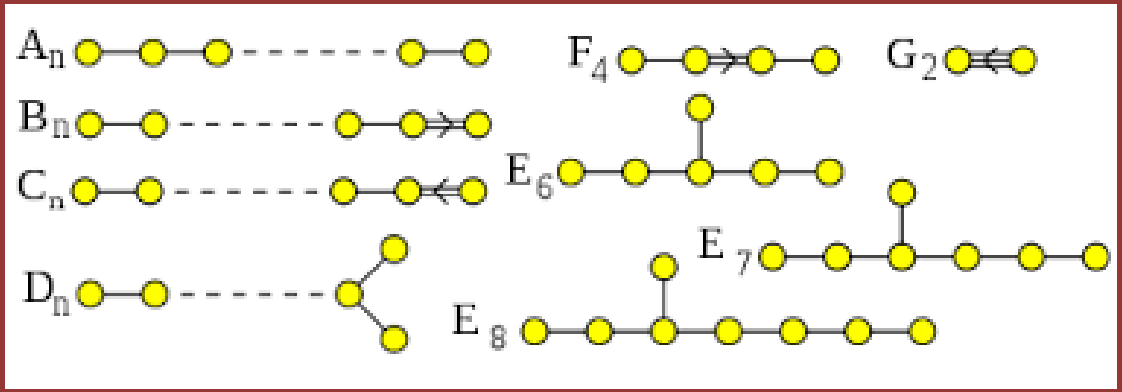
ADE graphs and simple Lie algebras

W. Killing (1888) classified simple Lie algebras

A_n: sl(n)
B_n: so(2n-1)
C_n: sp(2n)
D_n: so(2n)

E_6: dim = 78
E_7: dim = 133
E_8: dim = 248
F_4: dim = 52
G_2: dim = 14

Simple Lie algebras ⇨ Root systems ⇨ Dynkin diagrams



Lie groups, symmetric spaces, reflection groups

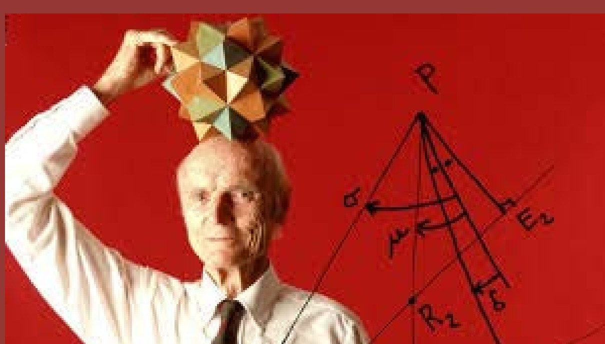
Simple Lie algebras ⇒ Lie groups ⇒ Symmetric spaces

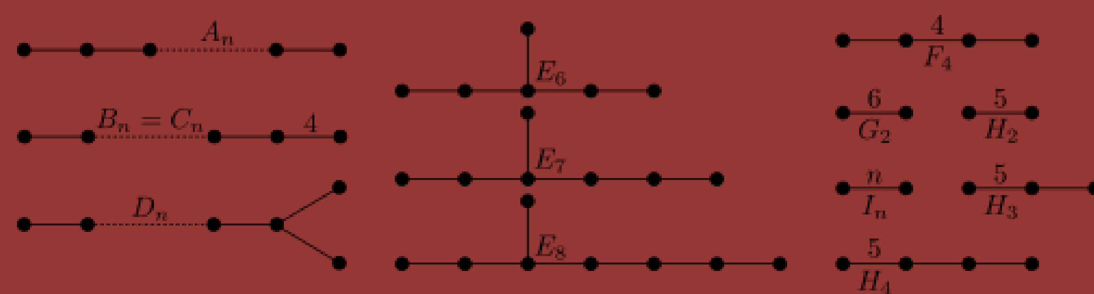
⇕

Weyl groups (generalizations of S_n)

⇕

Coxeter groups (generated by reflections)





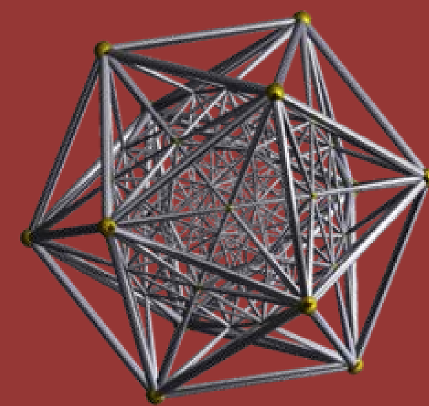
Two other occurrences of ADE graphs

Pierre Gabriel (1972): quivers of finite type and indecomposable representations

John McKay (1979): ADE graphs correspond to finite subgroups of SU(2) / unit quaternions

The binary polyhedral groups are:

- A_n: binary cyclic group of an (n + 1)-gon, order 2n
- D_n: binary dihedral group of an n-gon, <2,2,n>, order 4n
- E_6: binary tetrahedral group, <2,3,3>, order 24
- E_7: binary octahedral group, <2,3,4>, order 48
- E_8: binary icosahedral group, <2,3,5>, order 120



The 600-cell (E_8)

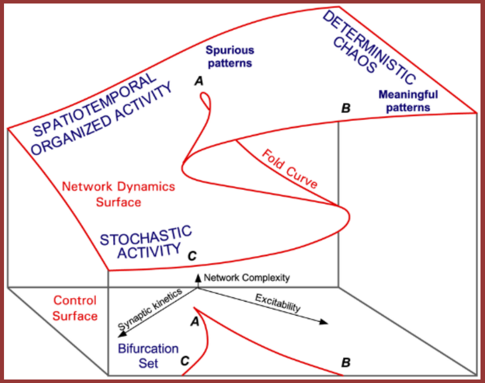
$$1^2+2^2+3^2+4^2+5^2+6^2+4^2+2^2+3^2 = 120$$

Many other occurrences of ADE graphs !!

Von Neumann algebras and II-1 factors

Conformal field theory and Wess-Zumino-Witten models (fusion rule algebras) String theory!

Catastrophe theory



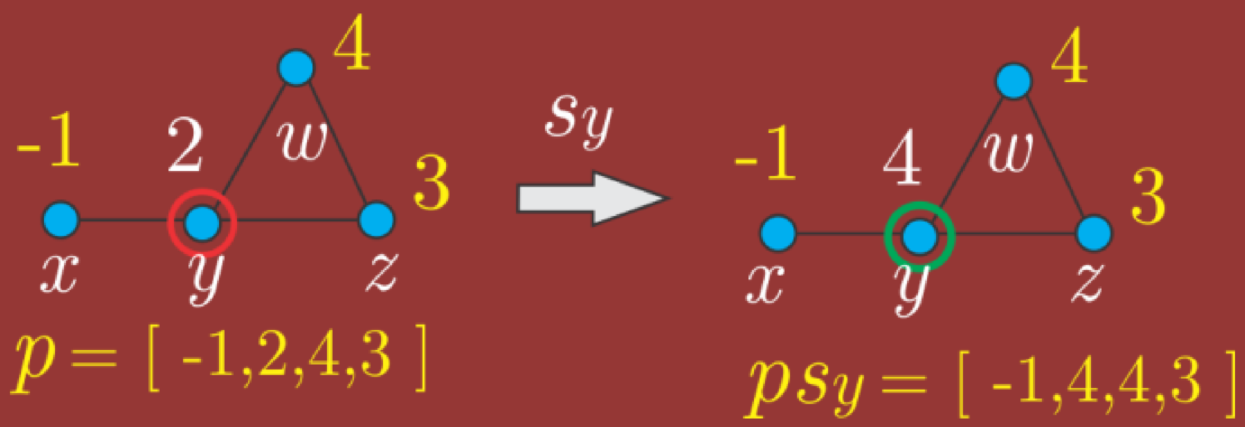
Simple singularites of holomorphic functions (V I Arnold)

Combinatorics!!

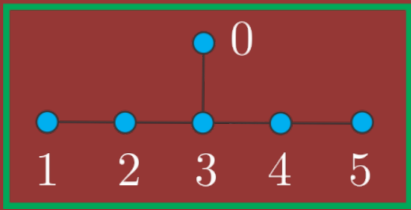
The Mutation Game

X = simple, connected graph. A population on X is an integer valued function on the vertices.

P(X) = populations on X

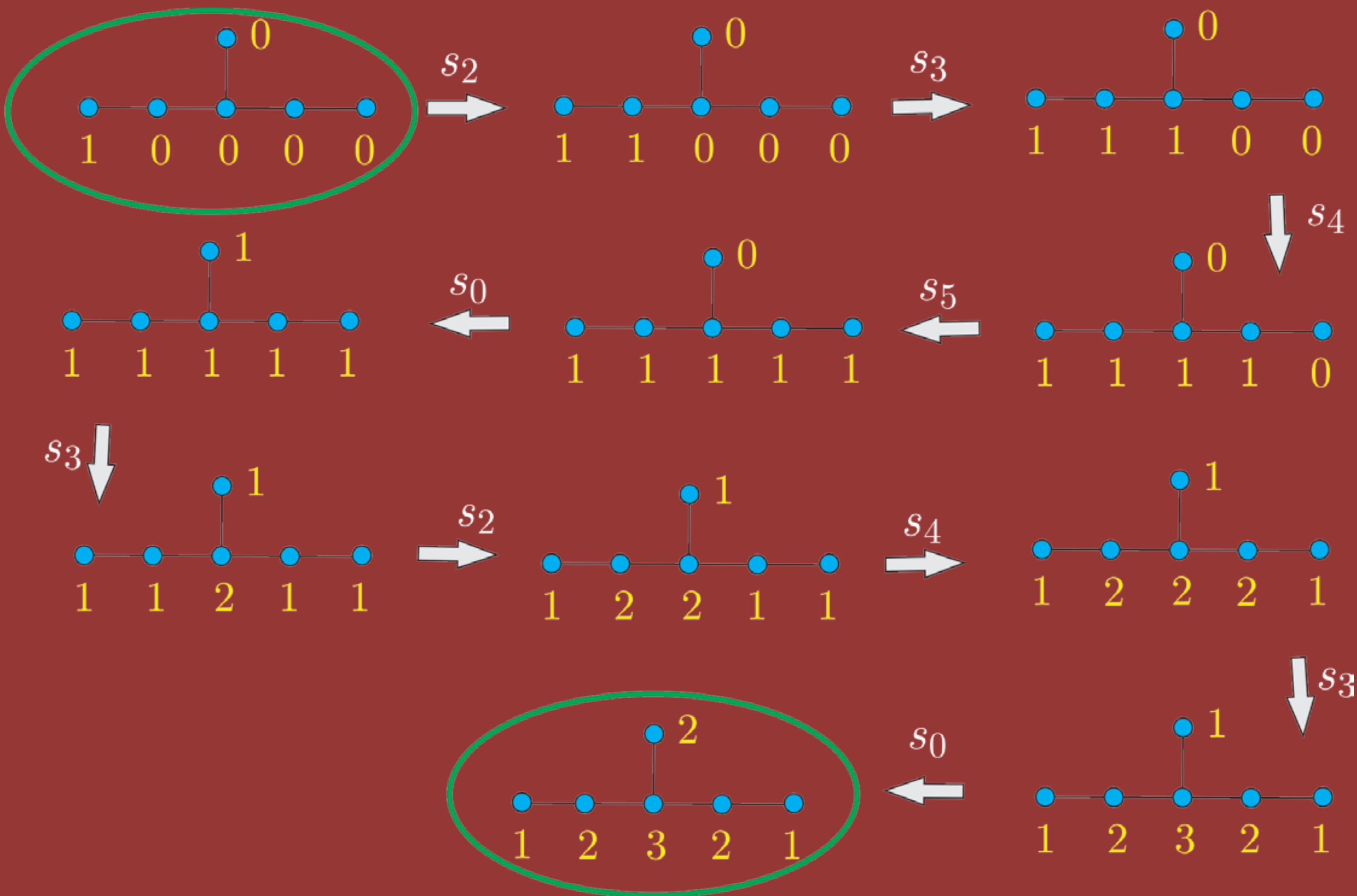


The mutation s_y at the vertex y: fixes all population values except that at y, which gets replaced by its negative plus the sum of its neighbours.



E6 mutation sequence

From a singleton population to the maximal population

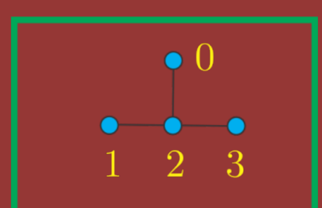


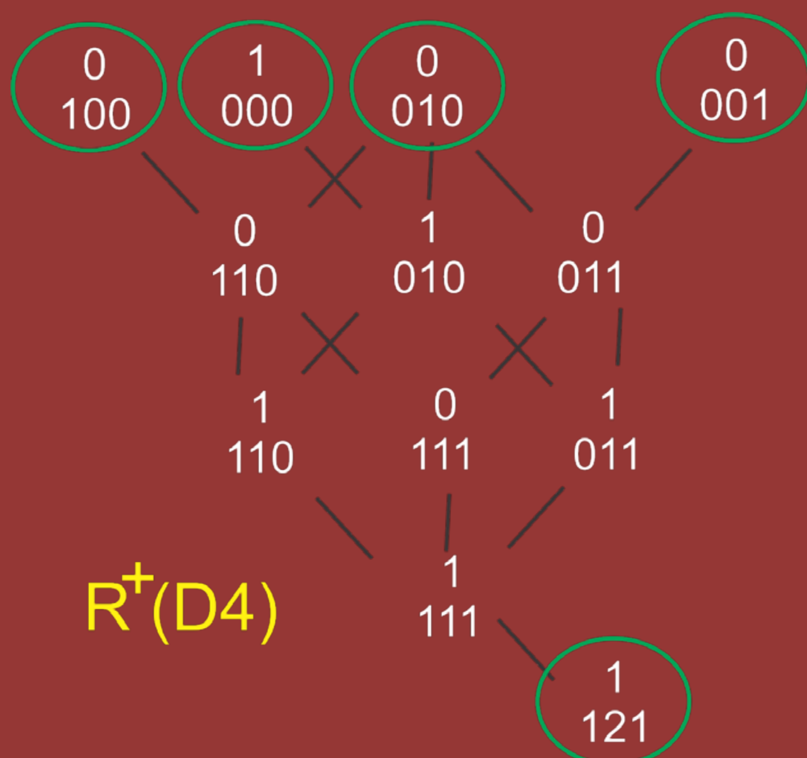
Root Populations / roots

A root population is a population obtained from a singleton population by applying any sequence of mutations. R(X) denotes the root populations of the graph X.

$$R(D4) = R^{+}(D4) + R^{-}(D4)$$

Where a root is positive if all its entries are positive (>=0).

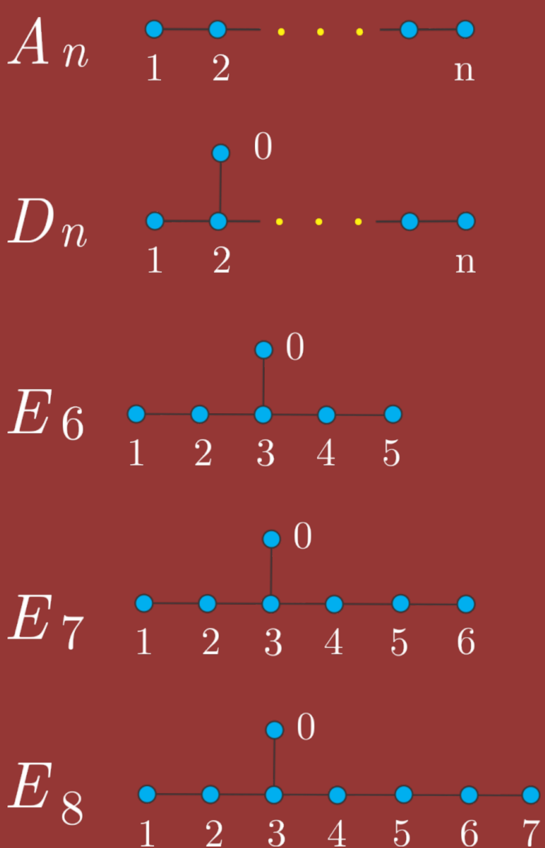




$R^{+}(D4)$

ADE Graphs

Theorem: R(X) is finite precisely when X is an ADE graph, i.e. in this following list:



These sets R(X) turn out to be the irreducible (simply laced) root systems studied by E. Dynkin.

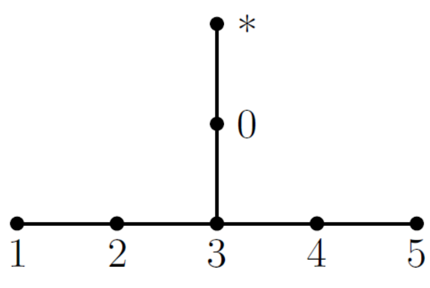
These are sets of vectors in a Euclidean space satisfying symmetry wrt reflections in hyperplanes

Proof:

1: If X is ADE then R(X) is finite (enumerate them!)

2. If X is not ADE, then it contains an ADE~ subgraph

3. Show that if X is ADE~ then R(X) is unbounded: use the Perron Frobenius vector which is unchanged by mutations, for example for E6:



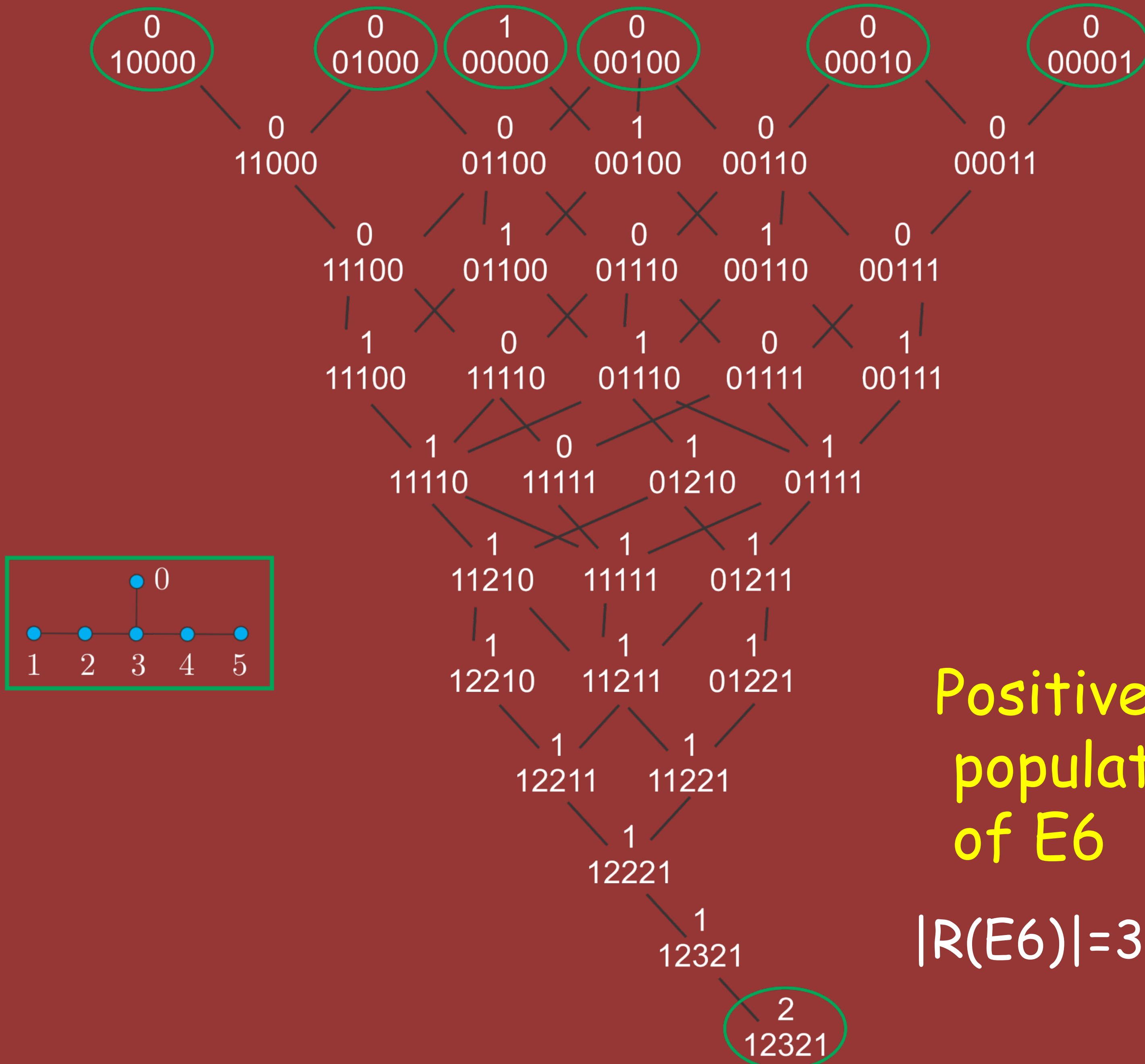
$$\delta_3 s(2,4,0,1,5,*,3,2,4,0,3) = \delta_3 + \alpha_0$$

An important conjecture

The Mutation Fact / Conjecture: For any simple connected graph X, R(X) is always the disjoint union of positive and negative roots.

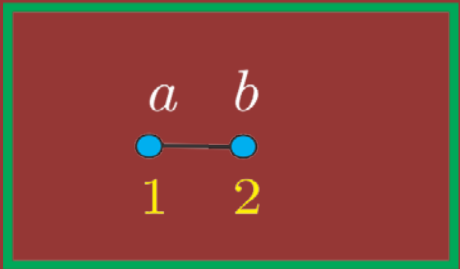
This is known as a consequence of the theory of Coxeter groups, generated by reflections. (Personal communication with Bob Howlett). However we do not have a combinatorial proof/ understanding.

A restatement: a root population can never have both strictly positive and strictly negative entries, for any graph X !!



Positive root populations of E6

$|R(E6)|=36+36=72$



A2: The two-dimensional mutation representation of W(A2) = S_3 = ‹ s1 , s2 ›

$$
\begin{tikzcd}
 & & e \begin{pmatrix}1&0\\0&1\end{pmatrix} & & \\
 & & (a , b) \arrow[dl, "s1"'] \arrow[dr, "s2"] & & \\
\begin{pmatrix}-1&0\\1&1\end{pmatrix} & (-a+b , b) \arrow[d, "s2"'] & & (a , a-b) \arrow[d, "s1"] & \begin{pmatrix}1&1\\0&-1\end{pmatrix} \\
\begin{pmatrix}-1&-1\\1&0\end{pmatrix} & (-a+b , -a) \arrow[dr, "s1"'] & & (-b , a-b) \arrow[dl, "s2"] & \begin{pmatrix}0&1\\-1&-1\end{pmatrix} \\
 & & (-b , -a) & & \\
 & & \begin{pmatrix}0&-1\\-1&0\end{pmatrix} & &
\end{tikzcd}
$$

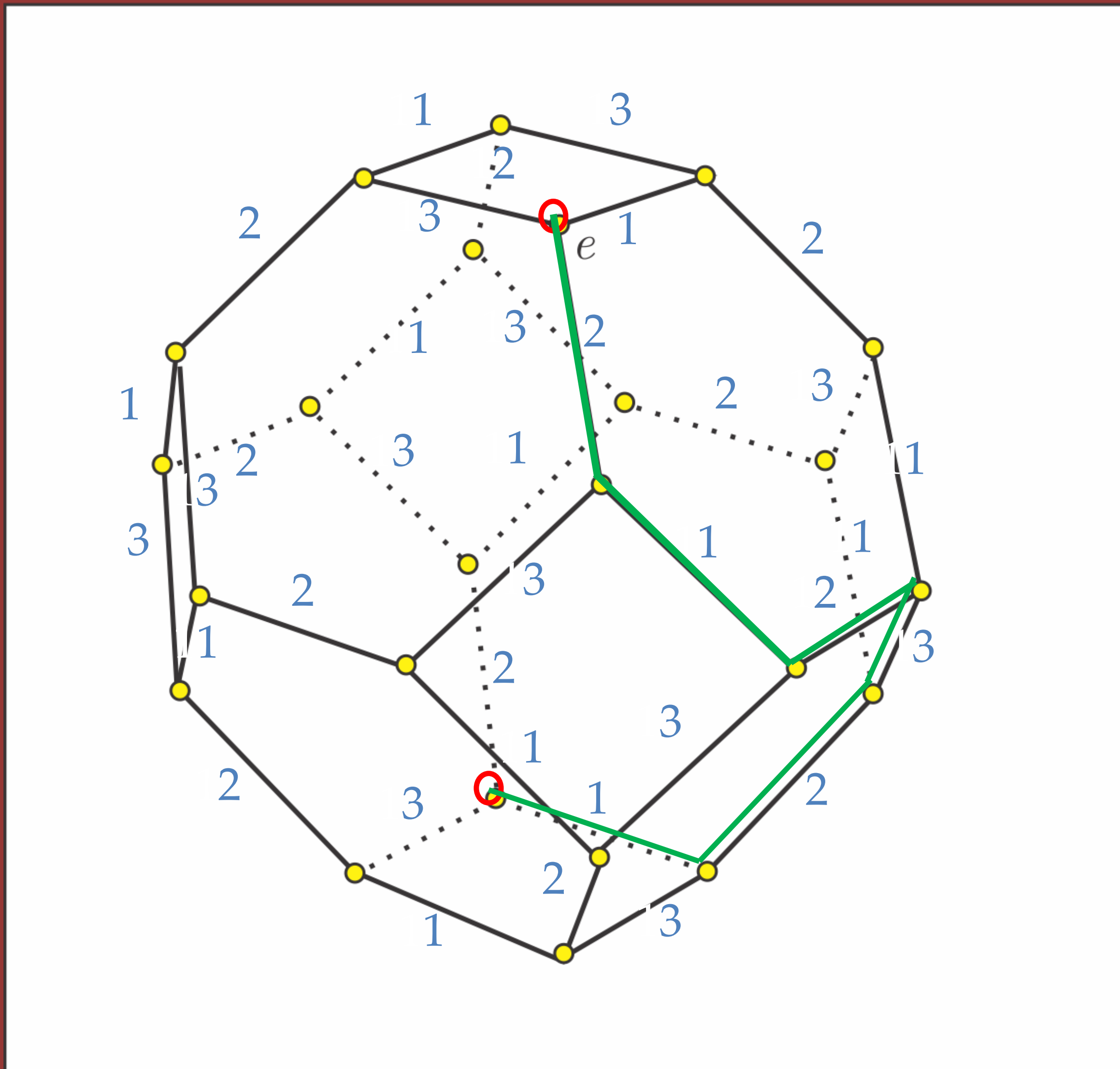
Reflection relation

S1^2 = s2^2 = 1

Braid relation

s1 s2 s1 = s2 s1 s2

A3: The three-dimensional mutation representation of W(A3) = S_4 = < s1 , s2, s3 > and the Permutahedron



(a , b , c)

(a , a-b+c , c)

(-b+c, a-b+c, c)

(c-b, -a+c, c)

(c-b, -a+c, -a)

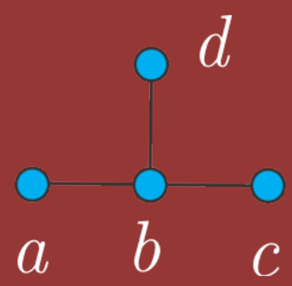
(c-b, -b, -a)

(-c, -b, -a)

Longest word in W

The Tits Quadratic Form

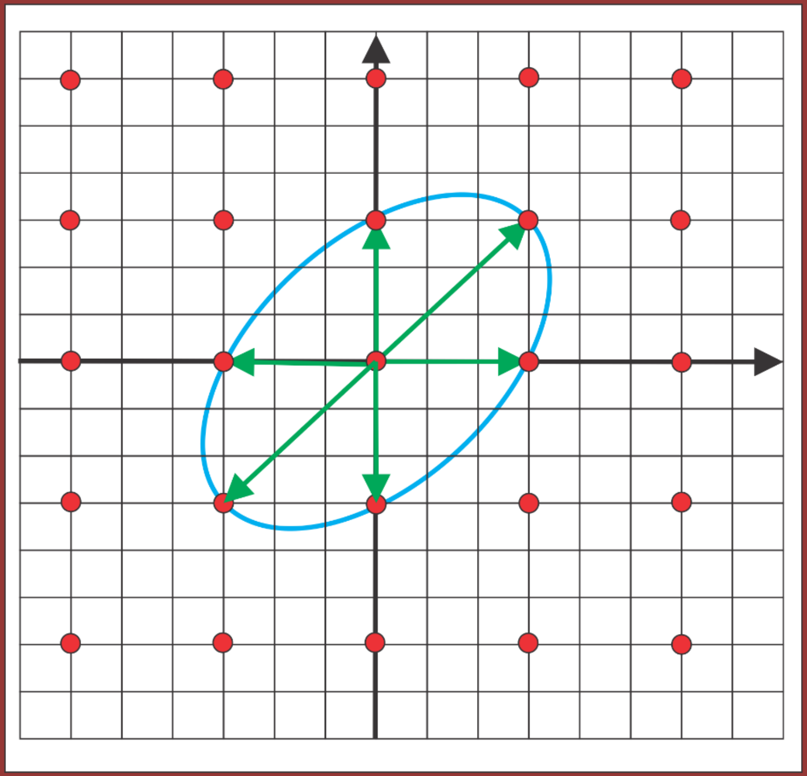
Define a symmetric bilinear form on P(X) = populations on X via the symmetric matrix C = 2I-A [Cartan matrix] where A is the adjacency of the graph.

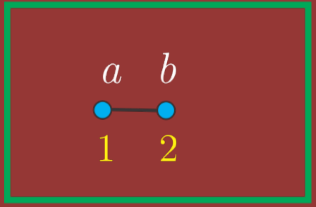


$$Q(a,b,c,d) = 2a^2 + 2b^2 + 2c^2 + 2d^2 - ab - bc - bd$$

Theorem: The mutations s_x are isometries with respect to the Tits quadratic form. So W = < s_x > is a group of isometries.

A2 lattice, sphere Q(v)=2 and root system





Q((a,b))=2a^2+2b^2-ab

Rational Trigonometry

A symmetric bilinear form gives geometry! The algebraic approach forgets about distances and angles, and uses quadrance and spread!

$$s\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = \frac{\left(x_1 y_2 - x_2 y_1\right)^2}{\left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right)}$$

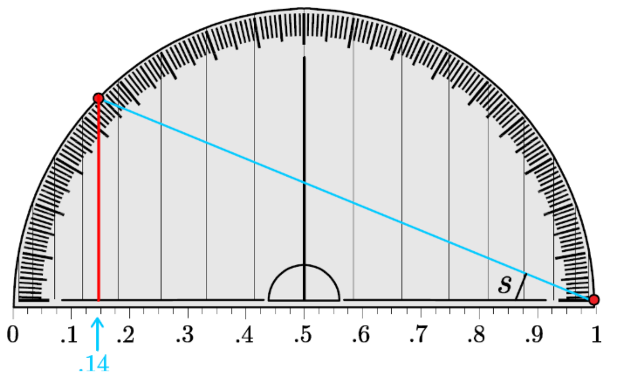
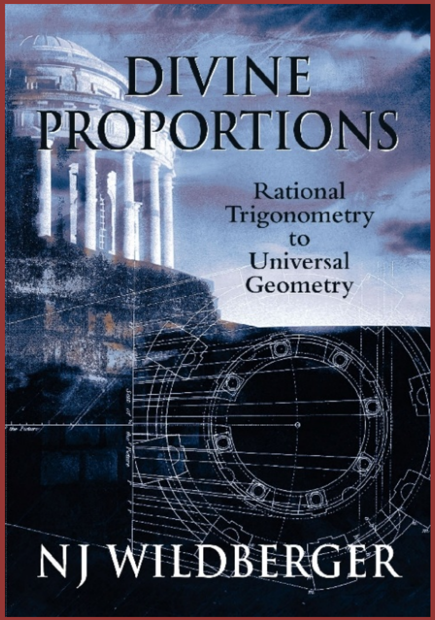


Figure: Paul Miller's spread protractor: $s = 0.14$



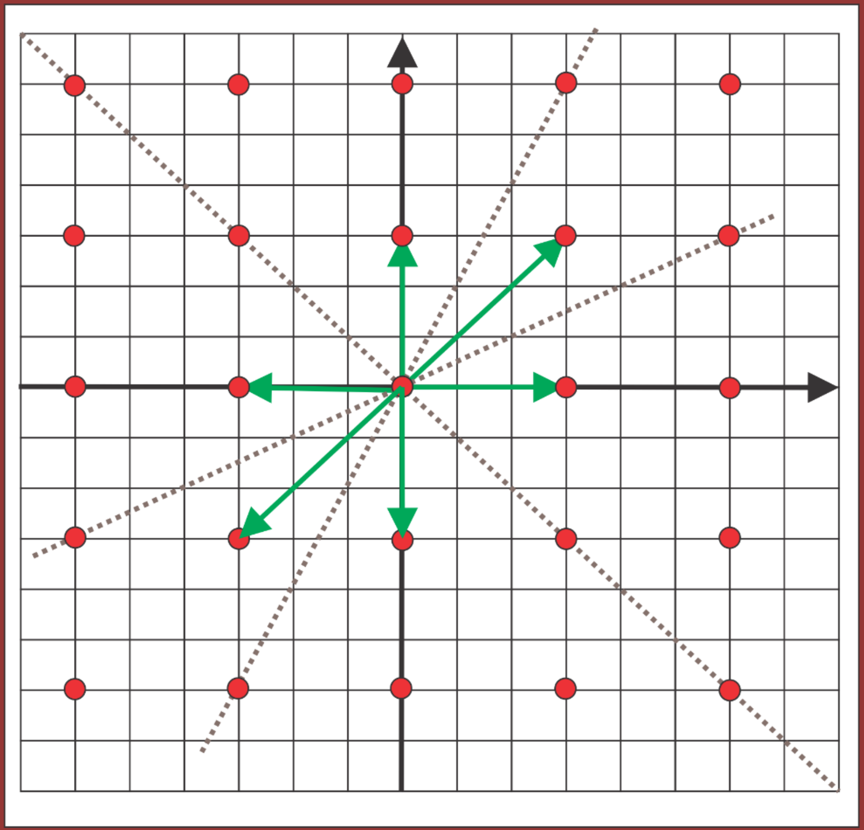
Extends to general quadratic forms!

Old Babylonian Trigonometry

Plimpton 322 from 1800 B.C.E. is the world's first trigonometric table: using ratio-based trigonometry!

[Plimpton 322 is Babylonian exact sexagesimal trigonometry, Mansfield D., Wildberger N.J., 2017 Historia Mathematica]

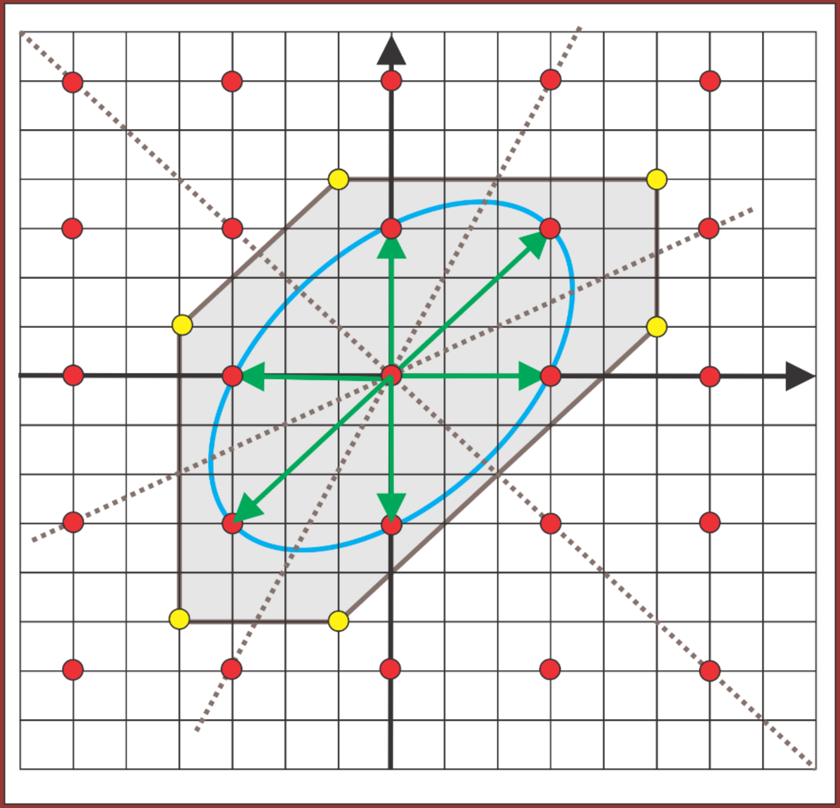
Root systems (generalized)



A generalized (simply - laced) root system is a type of vector in an inner product space, each with the same quadrance, invariant under reflections in any associated perpendicular hyperplane, with reflections given by integral multiples of root vectors.

Theorem: R(X) for any graph X is a generalized simply- laced root system. It is finite precisely when X is ADE.

Root hyperplanes, Weyl chambers and orbits



A Weyl group W orbit for A2

Size of root populations and Weyl groups W = < s_x >

$|R(A_n)| = n^2 + n$

$|W(A_n)| = (n + 1)!$

$|R(D_n)| = 2n^2 - 2n$

$|W(D_n)| = 2^n . n!$

$|R(E6)| = 36 + 36 = 72$

$|W(E6)| = 51{,}840$

$|R(E7)| = 63 + 63 = 126$

$|W(E7)| = 2{,}903{,}040$

$|R(E8)| = 120 + 120 = 240$

$|W(E8)| = 696{,}729{,}600$

•

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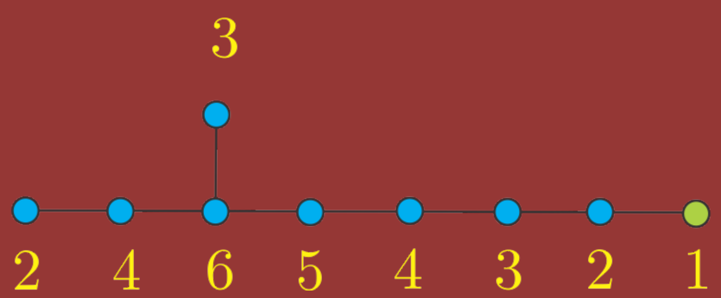
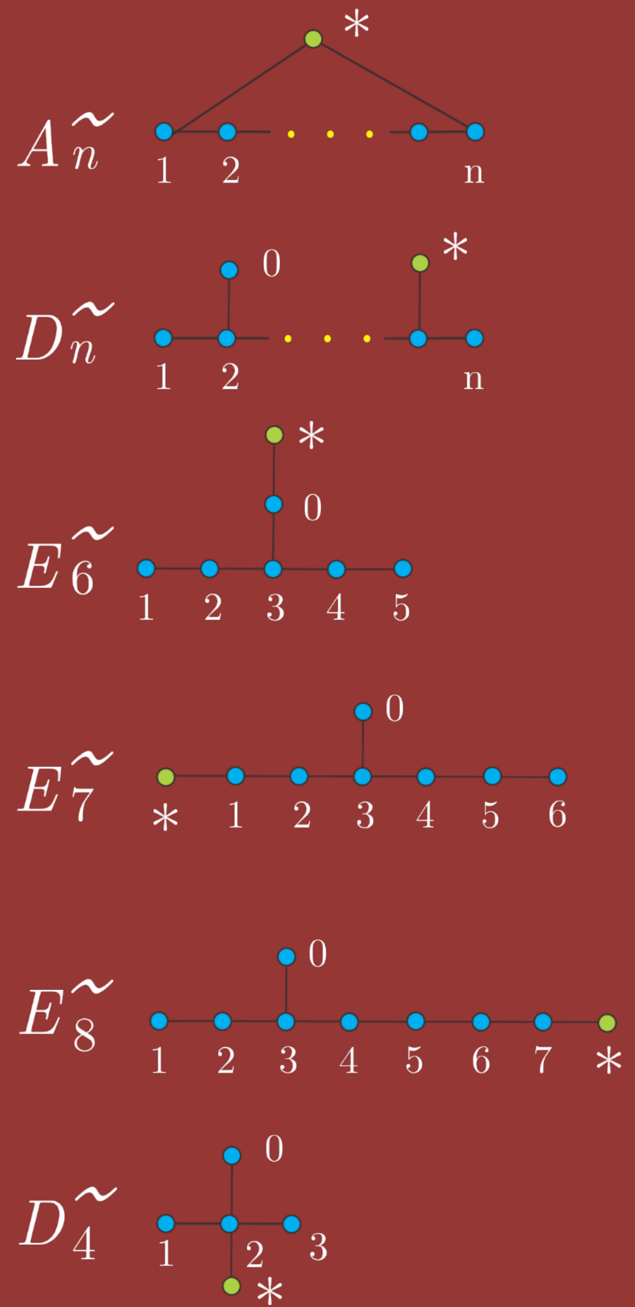
ADE ~ Graphs

Theorem: The Tits quadratic form is degenerate precisely when X is an ADE~ graph.

Then spectral radius (X)= 2, and a Perron Frobenius vector has quadrance Q(v) = 0



Remove the ~ node, and you get the maximum root population on the associated ADE graph

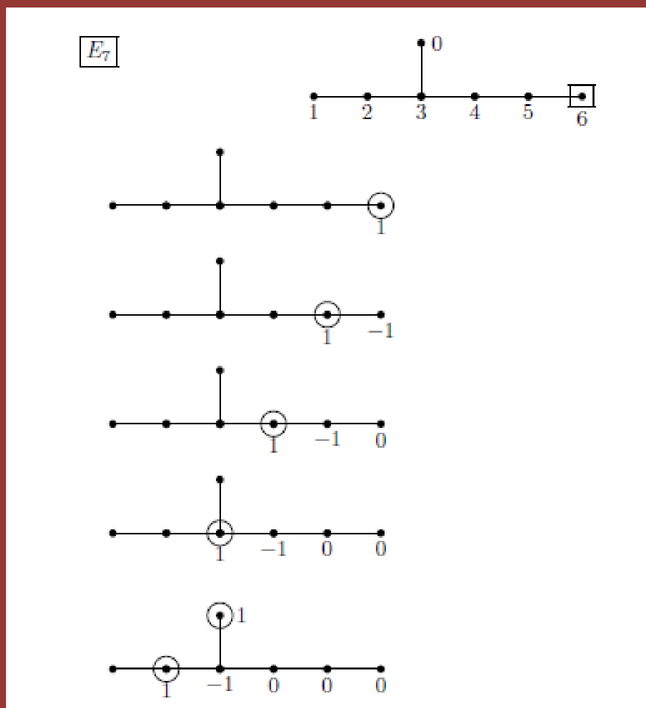


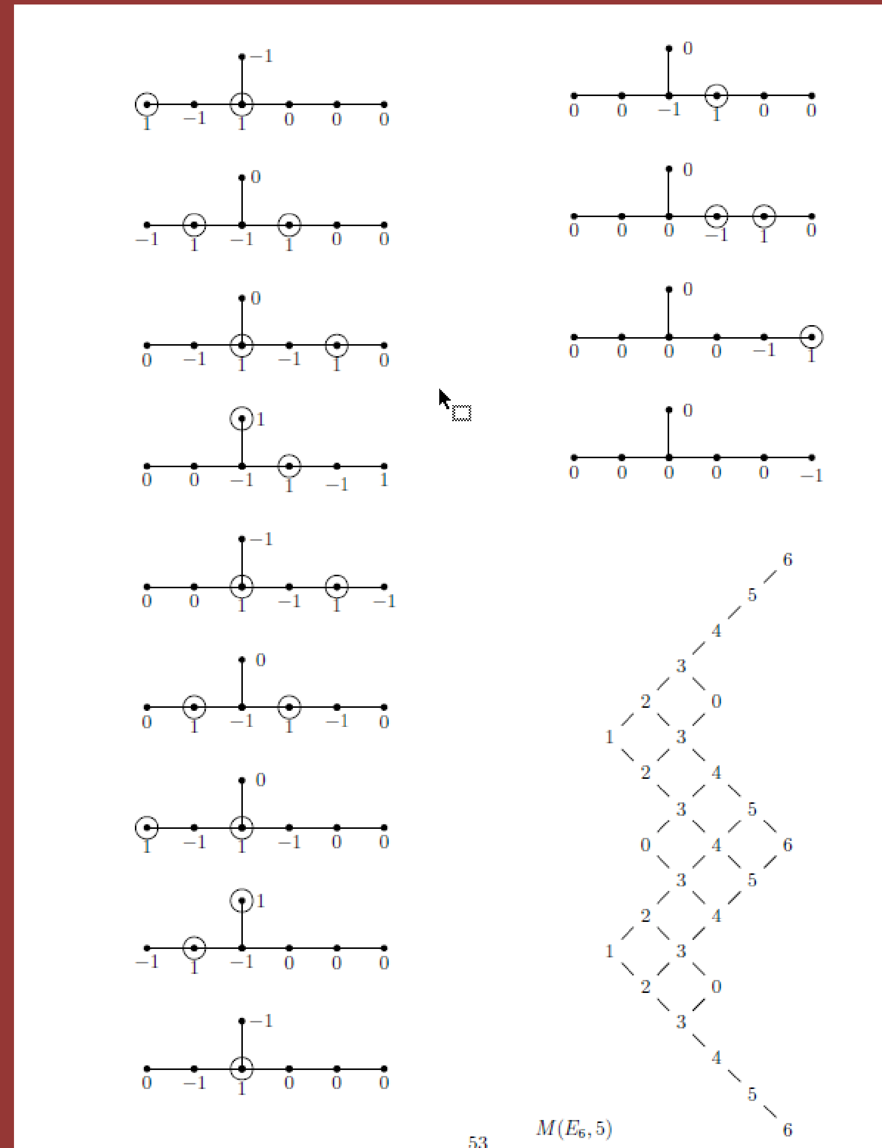
Remarkable lattices

In the case of an ADE graph, P(X) is a (Euclidean) geometric lattice since the Tits quadratic form is positive definite. But these lattices also have remarkable properties!

Dim	1	2	3	4	5	6	7	8	> 8
Best Lattice Packing	A1	A2	A3	D4	D5	E6	E7	E8	??
Largest Kissing Number	A1	A2	A3	D4	D5	E6	E7	E8	…
Number of sphere neighbours	2	6	12	24	40	72	126	240	…

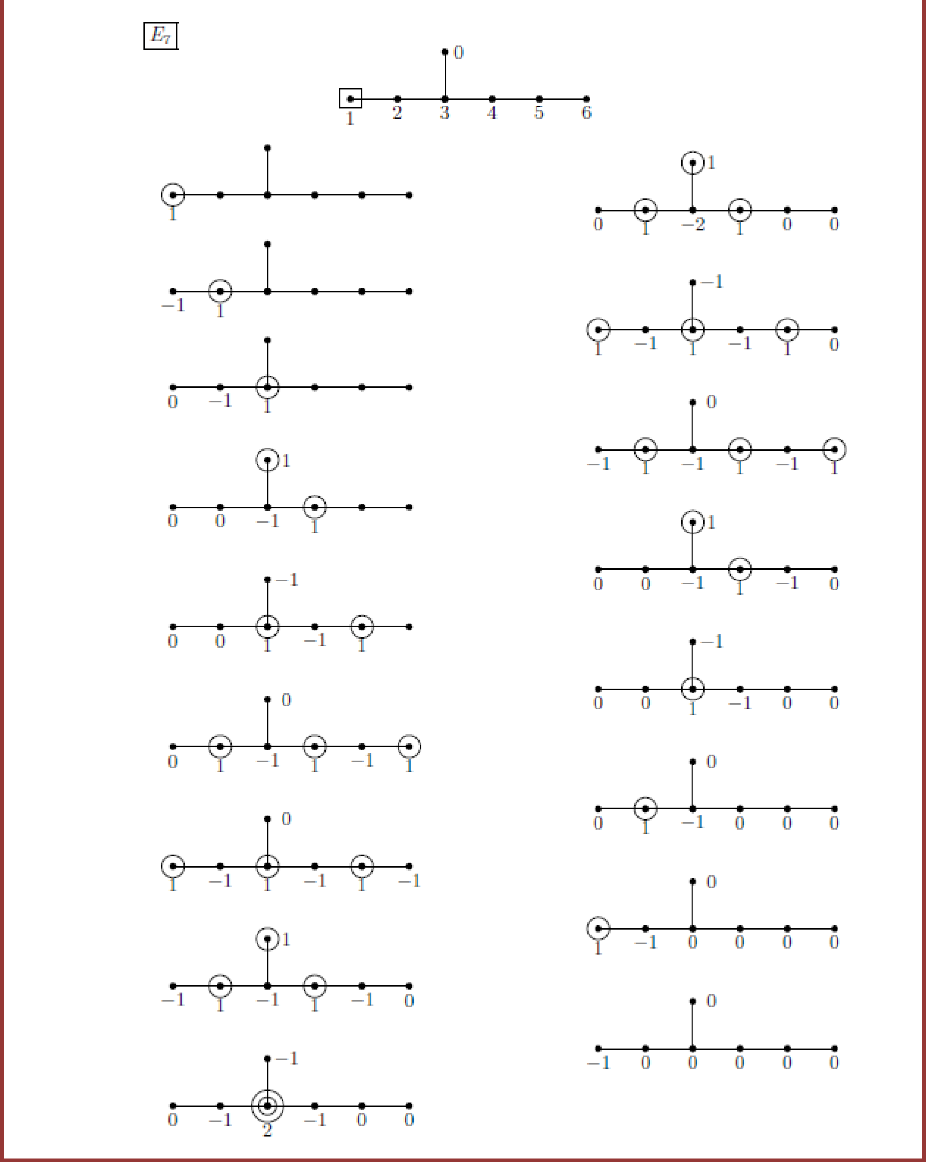
An E7 Mutation Sequence

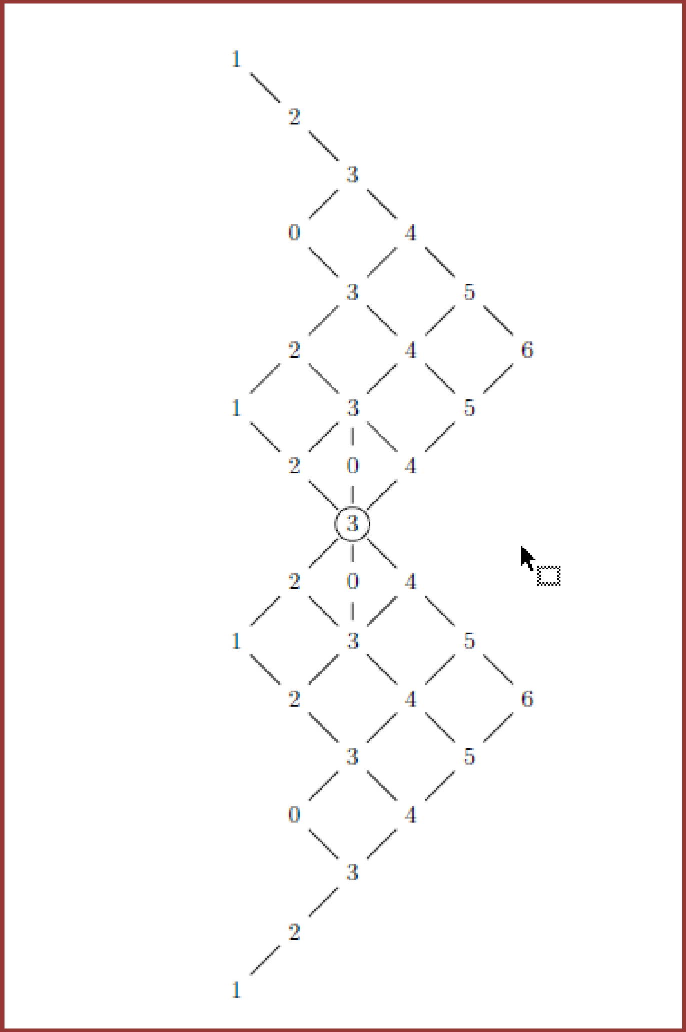




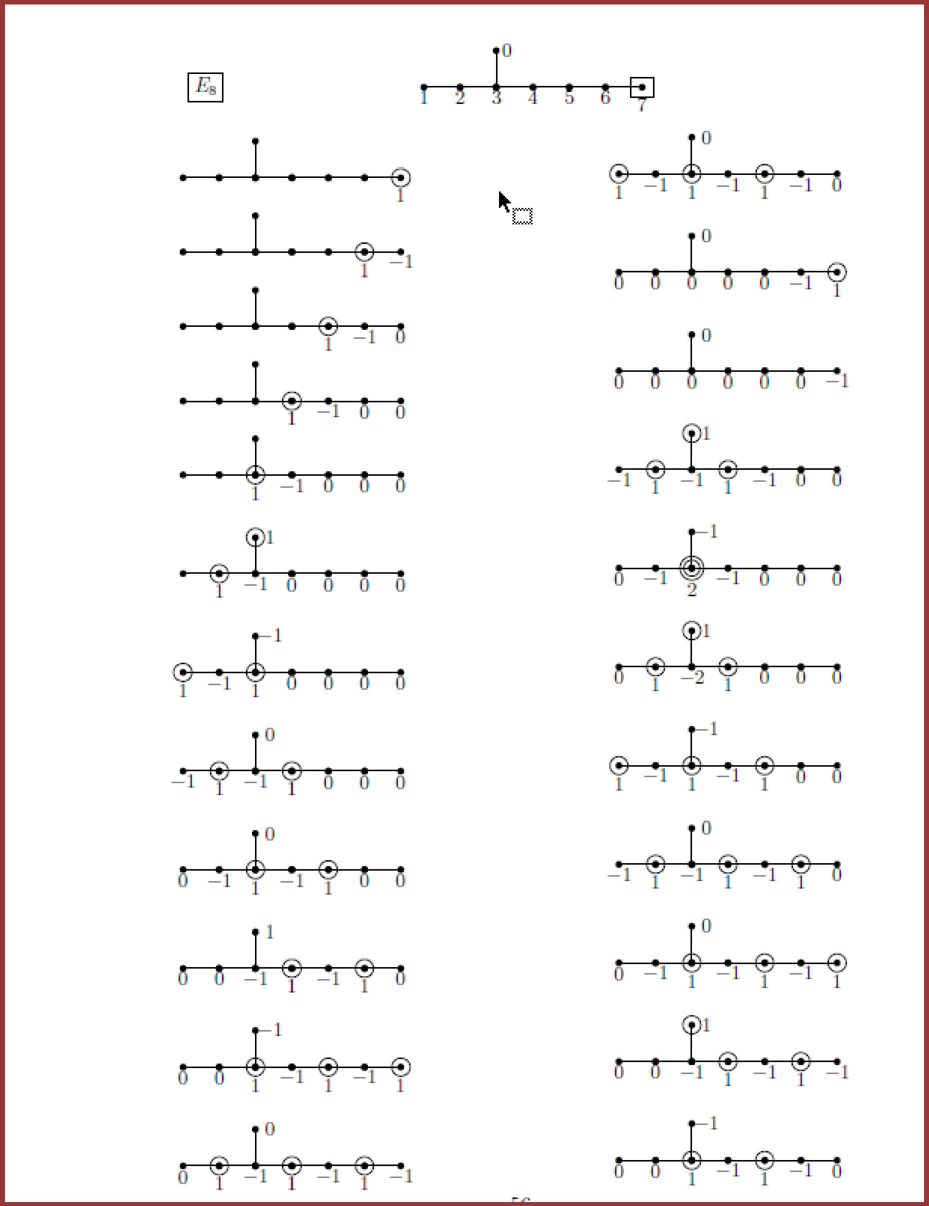
And an associated mutation frame... the swallow!

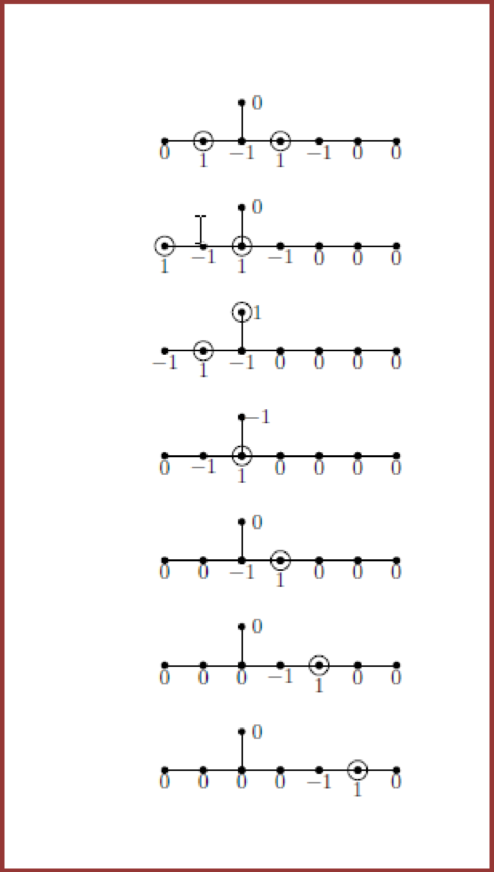
Another E7 Mutation Sequence



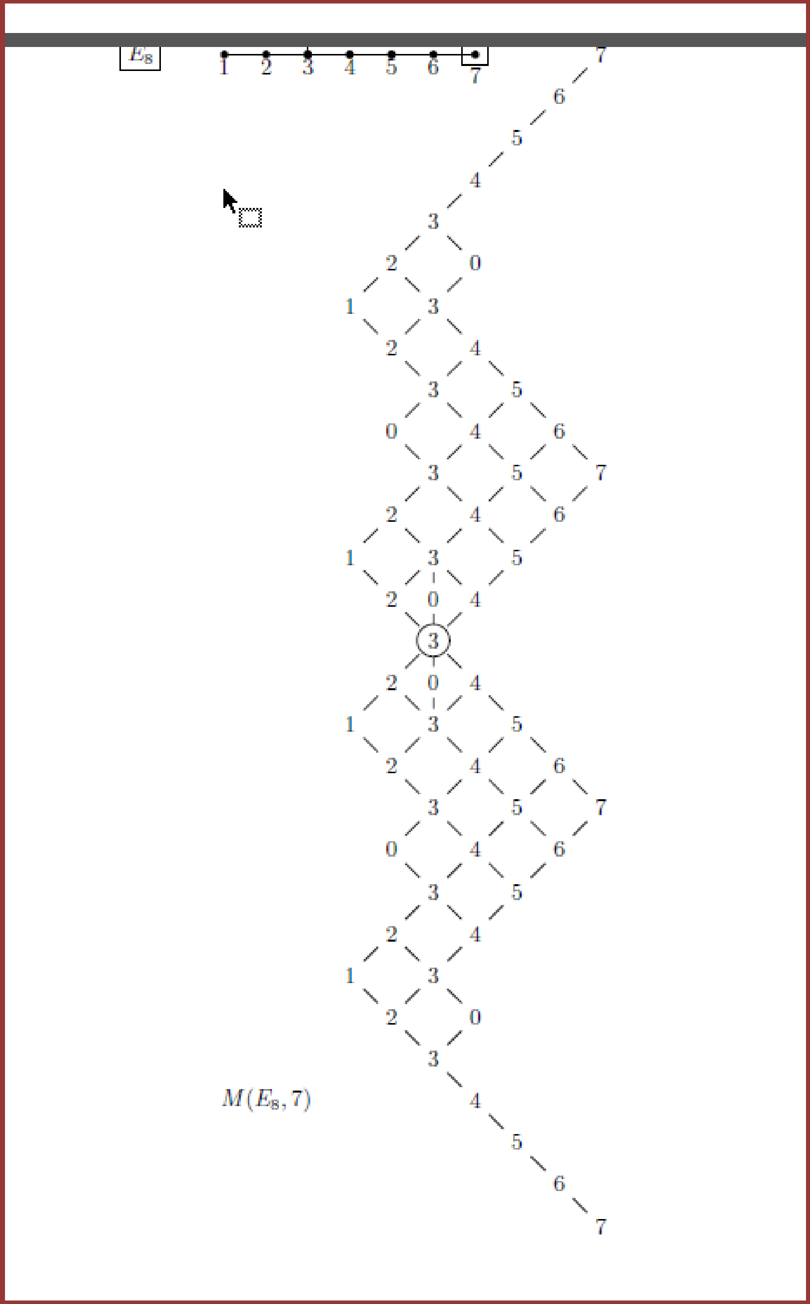


An E8 Mutation Sequence





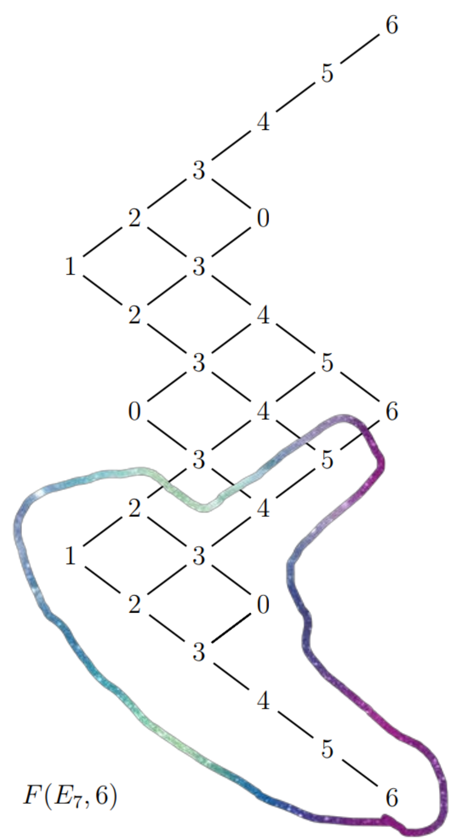
The associated E8 Mutation Frame



This is a pomset: a partially ordered multiset!

The swallow: the E7 X-heap

Theorem 2.1 *Let X be a simple graph for which there exists a maximal neighbourly X-heap F which is two-neighbourly. Then X is one of the graphs $A_n, n \geq 1$, D_n, $n \geq 4$, E_6 or E_7. There are exactly n such X-heaps for A_n, three for D_n, two for E_6 and one for E_7.*



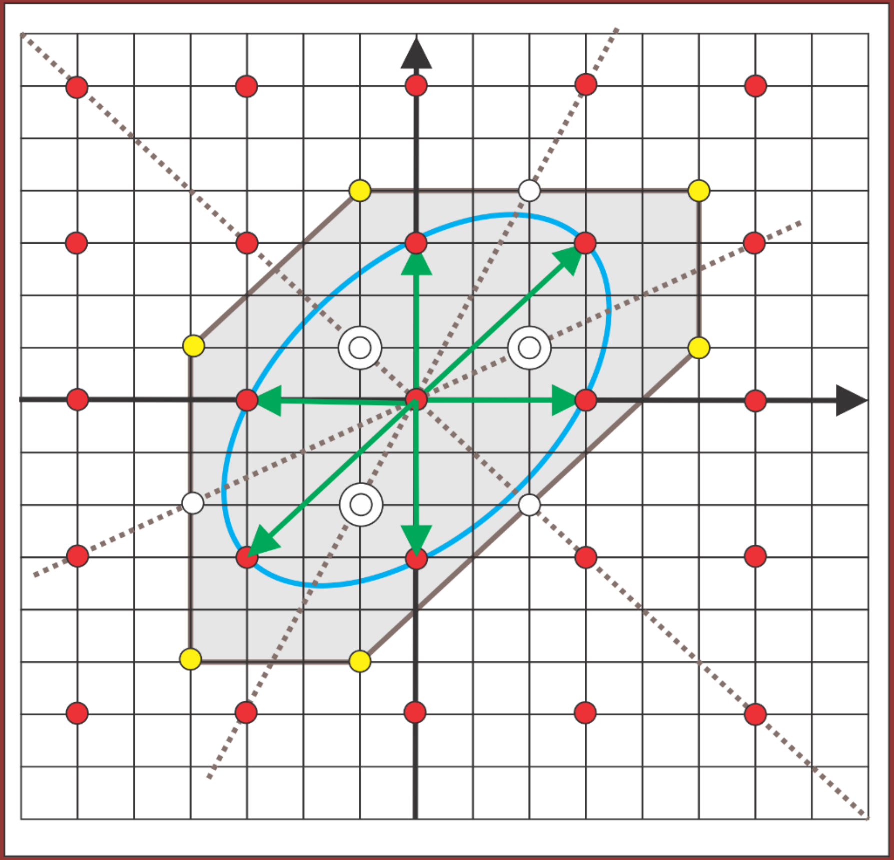
$F(E_7, 6)$

In "A Combinatorial Construction of simply-laced Lie algebras" (2003), I show how to construct ADE Lie algebras except E8 through minuscule representations via spaces of ideals on such heaps.

For E7 this gives the smallest 56 dim representation.

In another paper I give a similar realization of the 14 dim rep of G2.

Lie algebra representations and weights



15 dim representation of sl(3) or su(3)

Mutations, Root systems and related heaps/lattices/pomsets on general graphs X:

A huge area of potential investigation!

Thanks for listening, and many thanks to the organizers of G2 R2!

Join the Algebraic Calculus One course:
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