POLARITIES OF PROJECTIVE PLANES AND RELATED GRAPHS

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Two sides of the same coin

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ALGEBRA GEOMETRY

Two sides of the same coin

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"Should you just be an algebraist or a geometer?" is like saying "Would you rather be deaf or blind?"

***M. Atiyah**, Mathematics in the 20th century*

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Question

What is the largest set of vectors in $V(3, \mathbb{K})$ such that no two are orthogonal?

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Recall : $(x_1, x_2, x_3) \perp (y_1, y_2, y_3) \Leftrightarrow x_1y_1 + x_2y_2 + x_3y_3 = 0.$

Delving into the question

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- The zero vector is irrelevant

  $\vec{0} \perp \vec{v} \quad \forall\, \vec{v} \in V(3, q)$

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- The zero vector is irrelevant

  $\vec{0} \perp \vec{v} \quad \forall\, \vec{v} \in V(3, q)$

- Orthogonality is defined up to scalar

  $\vec{v} \perp \vec{w} \Rightarrow \lambda\vec{v} \perp \vec{w} \quad \forall\, \vec{v}, \vec{w} \in V(3, q), \lambda \in \mathbb{F}_q$

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$$V(3, q)$$

vector lines

vector planes

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vector lines $\longrightarrow$ "points"

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$$V(3,q)$$

vector lines	$\longrightarrow$	“points”
vector planes	$\longrightarrow$	“lines”

Delving into the question

Question

What is the largest set of vector lines in $V(3,q)$ such that no two are orthogonal?

$V(3,q)$		$\mathrm{PG}(2,q)$
vector lines	$\longrightarrow$	points
vector planes	$\longrightarrow$	lines

Polarities of projective planes…

$V(3,q)$ $\mathrm{PG}(2,q)$

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point p

Polarities of projective planes…

$V(3, q)$ $\qquad\qquad$ $\mathrm{PG}(2, q)$

vector line p $\longleftarrow$ point p

Polarities of projective planes…

$$\begin{tikzcd}
V(3,q) & \mathrm{PG}(2,q) \\
\text{vector line } p \arrow[d] & \text{point } p \arrow[l] \\
\text{vector plane } p^{\perp} &
\end{tikzcd}$$

Polarities of projective planes...

$$
\begin{tikzcd}
V(3,q) & \mathrm{PG}(2,q) \\
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\text{vector plane } p^{\perp} \arrow[r] & \text{line } p^{\perp}
\end{tikzcd}
$$

Polarities of projective planes…

$$\begin{tikzcd}[column sep=huge, row sep=huge]
V(3,q) & \mathrm{PG}(2,q) \\
\text{vector line } p & \text{point } p \arrow[l] \arrow[d, red] \\
\text{vector plane } p^{\perp} \arrow[from=2-1, to=3-1] \arrow[r] & \text{line } p^{\perp}
\end{tikzcd}$$

Polarities of projective planes…

$$V(3,q) \qquad\qquad \mathrm{PG}(2,q)$$

$$\begin{tikzcd}
\text{vector line } \ell^{\perp} \arrow[r] & \text{point } \ell^{\perp} \\
\text{vector plane } \ell \arrow[u] & \text{line } \ell \arrow[l] \arrow[u]
\end{tikzcd}$$

Polarities of projective planes...

This map $\perp\colon \mathrm{PG}(2,q) \to \mathrm{PG}(2,q)$

- maps points to lines and vice versa
- is an involution
- reverses the incidence: $p \in \ell \iff p^\perp \ni \ell^\perp$

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Definition

A map satisfying these conditions is called a **polarity**.

. . . and related graphs

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Definition

The **polarity graph** is the graph with vertex set the points of $\mathrm{PG}(2, q)$ and

$$p \sim q \iff p \in q^{\perp}.$$

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...and related graphs

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$$p \sim q \iff p \in q^{\perp}.$$

Question

What is the largest independent set in the polarity graph?
I.e. try to determine its independence number α.

Back to the coin

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Hard to determine α exactly.

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$$\leq \alpha \leq$$

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Back to the coin

Hard to determine α exactly.

GEOMETRY $\leq \alpha \leq$ ALGEBRA

Theorem (Mubayi, Williford - 2004)

$\alpha = \Theta(q\sqrt{q})$

Mubayi and Williford's result

Let $q = 2^{2k+1}$,

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$$\frac{1}{2\sqrt{2}} q\sqrt{q} \leq \alpha \leq q\sqrt{q} + \sqrt{q} + 1$$

Mubayi and Williford's result

Let $q = 2^{2k+1}$,

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Theorem (Hobart, Williford - 2013)

For q even,

$$\alpha \leq q\sqrt{q} - q + \sqrt{q} + 1$$

Coherent configurations!

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Coherent configurations!

Sketch of proof.

1. construct the coherent configuration $(\Omega, \{R_i\}_{i=1}^{10})$, based on geometrical information
2. compute the irreducible representations $\{\Delta_j\}_{j=1}^{4}$
3. transfer algebraic knowledge to combinatorial information

Hobart subset bound

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Let $(\Omega, \{R_i\}_{i \in I})$ be a coherent configuration and $m_i = |R_i|$.

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Theorem (Hobart - 2009)

Consider a subset $X \subseteq \Omega$ and let $b_i = |(X \times X) \cap R_i|$. Define

$$D(X) = \sum_{i \in I} \frac{b_i}{m_i} A(R_i).$$

Then $\Delta_j(D(X))$ is positive semi-definite for any irreducible representation Δ_j.

Geometry time

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Theorem (M., Pavese, Storme - 201x)

$$\frac{1}{\sqrt{2}}q\sqrt{q} - q + \sqrt{\frac{q}{2}} + 1 \leq \alpha$$

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1. consider the symmetry of the problem, i.e. the group acting on the graph
2. find orbits that are cocliques
3. glue $\sqrt{q/2}$ of these orbits together in a good way

Unofficial announcement

FINITE GEOMETRY & FRIENDS

a Brussels summer school on finite geometry

17–21 June 2019

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Save the date!

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Thank you for your attention!