



POLARITIES OF PROJECTIVE PLANES AND RELATED GRAPHS

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Two sides of the same coin



Two sides of the same coin



ALGEBRA

GEOMETRY

Two sides of the same coin

ALGEBRA

GEOMETRY

“Should you just be an algebraist or a geometer?” is like saying “Would you rather be deaf or blind?”

M. Atiyah, Mathematics in the 20th century

Two sides of the same coin



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Question

What is the largest set of vectors in $V(3, \mathbb{K})$ such that no two are orthogonal?

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What is the largest set of vectors in $V(3, \mathbb{K})$ such that no two are orthogonal?

Recall : $(x_1, x_2, x_3) \perp (y_1, y_2, y_3) \Leftrightarrow x_1y_1 + x_2y_2 + x_3y_3 = 0$.

Delving into the question

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$$\vec{0} \perp \vec{v} \quad \forall \vec{v} \in V(3, q)$$

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What is the largest set of vectors in $V(3, q)$ such that no two are orthogonal?

Some considerations:

- ▶ The zero vector is irrelevant

$$\vec{0} \perp \vec{v} \quad \forall \vec{v} \in V(3, q)$$

- ▶ Orthogonality is defined up to scalar

$$\vec{v} \perp \vec{w} \Rightarrow \lambda \vec{v} \perp \vec{w} \quad \forall \vec{v}, \vec{w} \in V(3, q), \lambda \in \mathbb{F}_q$$

Delving into the question

Question

What is the largest set of **vector lines** in $V(3, q)$ such that no two are orthogonal?

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$V(3, q)$

vector lines

vector planes

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$PG(2, q)$

points

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Polarities of projective planes...

$$V(3, q)$$

$$\text{PG}(2, q)$$

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$V(3, q)$

$\text{PG}(2, q)$

point p

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vector line p ← point p

Polarities of projective planes...

$V(3, q)$

$PG(2, q)$

vector line p



point p



vector plane p^\perp

Polarities of projective planes...

$V(3, q)$

$PG(2, q)$

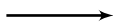
vector line p



point p



vector plane p^\perp



line p^\perp

Polarities of projective planes...

$V(3, q)$

$PG(2, q)$

vector line p



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vector plane p^\perp

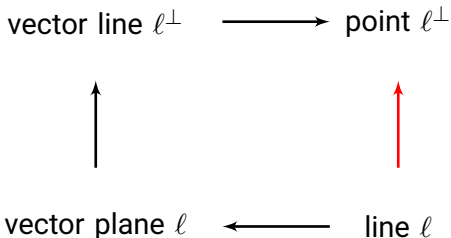


line p^\perp

Polarities of projective planes...

$V(3, q)$

$PG(2, q)$



Polarities of projective planes...

This map $\perp: \text{PG}(2, q) \rightarrow \text{PG}(2, q)$

- ▶ maps points to lines and vice versa
- ▶ is an involution
- ▶ reverses the incidence: $p \in \ell \iff p^\perp \ni \ell^\perp$

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Definition

A map satisfying these conditions is called a **polarity**.

... and related graphs



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Definition

The **polarity graph** is the graph with vertex set the points of $\text{PG}(2, q)$ and

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Question

What is the largest independent set in the polarity graph?
I.e. try to determine its independence number α .

Back to the coin



Back to the coin

Hard to determine α exactly.

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$$\underline{\leq} \alpha \underline{\leq}$$

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Theorem (Mubayi, Williford - 2004)

$$\alpha = \Theta(q\sqrt{q})$$

Mubayi and Williford's result

Let $q = 2^{2k+1}$,

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$$\frac{1}{2\sqrt{2}}q\sqrt{q} \leq \alpha \leq q\sqrt{q} + \sqrt{q} + 1$$

Mubayi and Williford's result

Let $q = 2^{2k+1}$,

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Theorem (Hobart, Williford - 2013)

For q even,

$$\alpha \leq q\sqrt{q} - q + \sqrt{q} + 1$$

Coherent configurations!



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Sketch of proof.

1. construct the coherent configuration $(\Omega, \{R_i\}_{i=1}^{10})$,
based on geometrical information
2. compute the irreducible representations $\{\Delta_j\}_{j=1}^4$
3. transfer algebraic knowledge to combinatorial information

Hobart subset bound



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Let $(\Omega, \{R_i\}_{i \in I})$ be a coherent configuration and $m_i = |R_i|$.

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Theorem (Hobart - 2009)

Consider a subset $X \subseteq \Omega$ and let $b_i = |(X \times X) \cap R_i|$. Define

$$D(X) = \sum_{i \in I} \frac{b_i}{m_i} A(R_i).$$

Then $\Delta_j(D(X))$ is positive semi-definite for any irreducible representation Δ_j .

$$\frac{1}{2\sqrt{2}}q\sqrt{q} \leq \alpha \leq q\sqrt{q} + \sqrt{q} + 1$$

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Theorem (M., Pavese, Storme - 201x)

$$\frac{1}{\sqrt{2}}q\sqrt{q} - q + \sqrt{\frac{q}{2}} + 1 \leq \alpha$$

Geometry time





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1. consider the symmetry of the problem, i.e. the group acting on the graph
2. find orbits that are cocliques
3. glue $\sqrt{q/2}$ of these orbits together in a good way

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Save the date!

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Thank you for your attention!