Non-residually Finite Direct Limit of the 2-Bridge Link Groups

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Open Problem

Is every hyperbolic group residually finite?

- Free groups are residually finite.
- It is commonly believed that a non-residually finite hyperbolic group exists.

1. - Residually finite groups, Hopfian groups
   - Hyperbolic groups, relatively hyperbolic groups
2. Construction of a non-residually finite group as a direct limit of 2-bridge link groups

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Definition(Residually Finite Groups)

A group G is said to be *residually finite* if for each $1 \neq g \in G$ there is a finite group H and a homomorphism $\varphi : G \to H$ such that $\varphi(g) \neq_H 1$.

Examples of non-residually finite groups:

- $\langle a, b \,|\, e_2 = e_3 = \cdots = 1 \rangle$, where $e_i = a^{-1}b^{-1}ab^{-i}ab^{-1}a^{-1}b^{i}a^{-1}bab^{-i}aba^{-1}b^{i}$ for $i = 2, 3, \ldots$. (Neumann, 1950)
- $\langle a, s, t \,|\, s^{-1}as = a^2, t^{-1}at = a^2 \rangle$. (Higman, 1951)
- $\mathrm{BS}(2,3) := \langle a, b \,|\, b^{-1}a^2ba^{-3} \rangle$ (Baumslag-Solitar, 1962)

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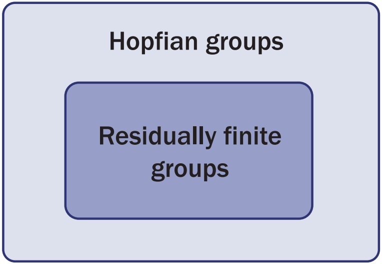
Definition(Hopfian Groups)

A group G is said to be *Hopfian* if every epimorphism $G \to G$ is an isomorphism.

Well-known Theorem(Mal'cev, 1940)

Every finitely generated residually finite group is Hopfian.

Any finitely generated non-Hopfian group is non-residually finite.



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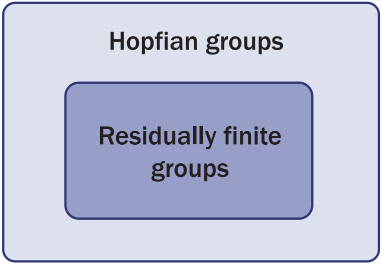

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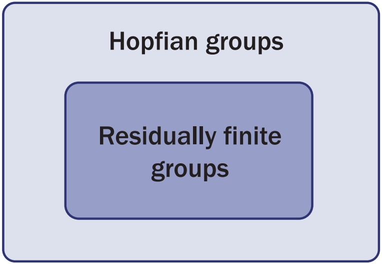
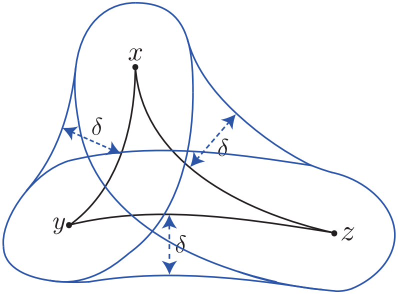


Let X be a geodesic metric space, and let $x, y, z \in X$. Denote by $[p, q]$ a geodesic segment with endpoints p and q.



(1) The union of $[x, y], [y, z], [z, x]$ is called the *geodesic triangle* with vertices x, y, z.

(2) A geodesic triangle is called δ-*slim* if there exists $\delta \geq 0$ such that the union of the closed balls $B_\delta([y, z]) \cup B_\delta([z, x])$ contains $[x, y]$, and similarly for $[y, z]$ and $[z, x]$.

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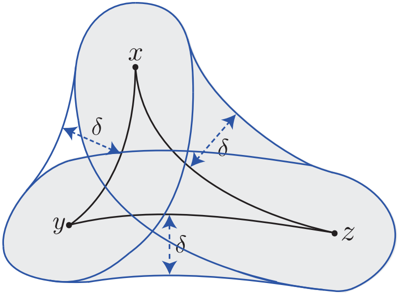

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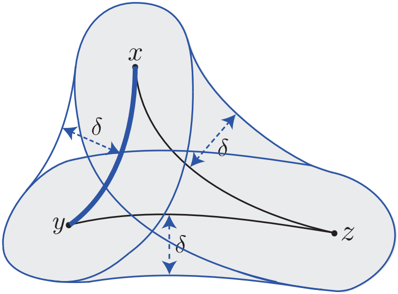

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Definition(δ-Hyperbolic)

A geodesic metric space X is called δ-*hyperbolic* if every geodesic triangle of X is δ-slim for some $\delta \geq 0$.

Definition(Hyperbolic Groups)

A group G is said to be *hyperbolic* if every Cayley graph of G is δ-hyperbolic for some $\delta \geq 0$.

- Every hyperbolic group is finitely presented. (Gromov, 1987)
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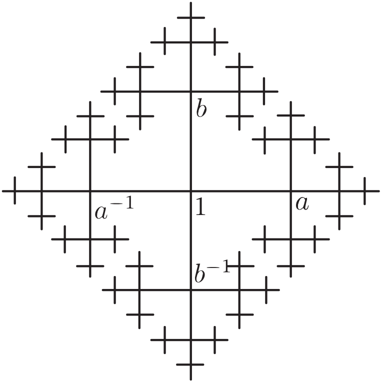

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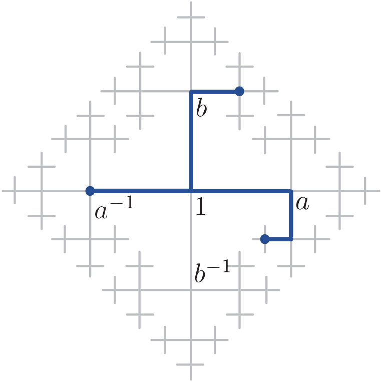

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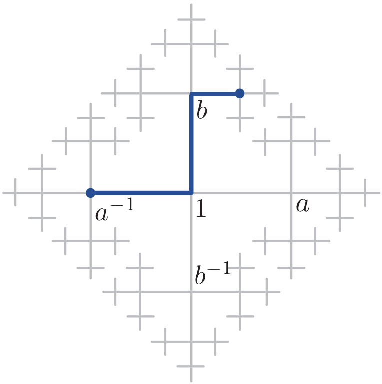

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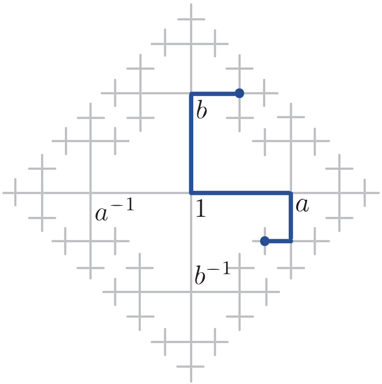

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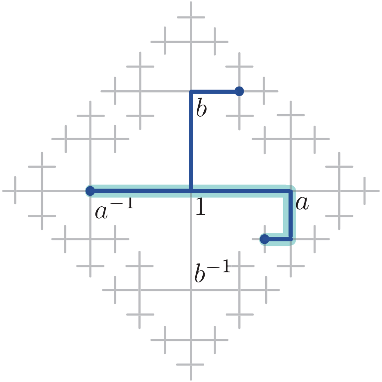
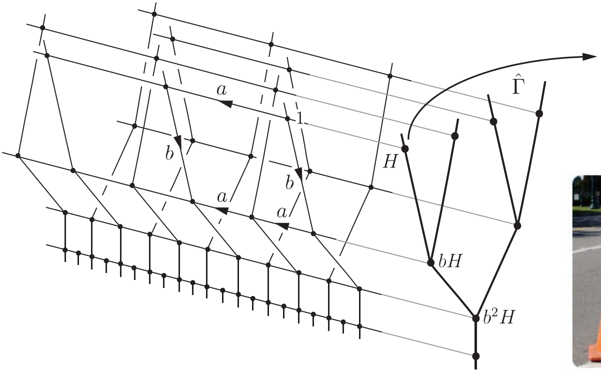


Definition(Relatively Hyperbolic Groups, Farb)

Let G be a group generated by a finite set X and $H \leq G$. Then G is called *hyperbolic relative to* H if the coned-off Cayley graph of G w.r.t. H is hyperbolic.



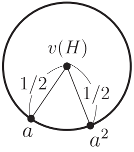


Figure: The coned-off Cayley graph of $\mathrm{BS}(1,2)$ with the subgroup $H = \langle a \rangle$.

Definition(Relatively Hyperbolic Groups)

Let G be a group generated by a finite set X and $H \leq G$. Then G is called *hyperbolic relative to* H if the coned-off Cayley graph of G w.r.t. H is hyperbolic, and G satisfies the Bounded Coset Penetration property.

Definition(Relatively Hyperbolic Groups, Osin)

Let $G = \langle X, H \mid R = 1, R \in \mathcal{R} \rangle$ be a *relatively finitely presented* ($X, \mathcal{R}$ are finite) *group* w.r.t. H. Here, $\mathcal{R} = \mathrm{Ker}\epsilon$, where $\epsilon : H * F(X) \to G$ is a canonical epimorphism.

If $w =_G 1$ then there exists a representation of w as

$$w \equiv \prod_{i=1}^{k} f_i^{-1} R_i f_i,$$

where $f_i \in H * F(X)$ and $R_i \in \mathcal{R}$.

For every word w in $X^{\pm 1} \cup H$ satisfying $w =_G 1$,

$$\delta_{G,H}^{rel}(n) = \max\{\text{the minimal number of conjugates in } w \mid |w| \leq n\}$$

is called the *relative Dehn function* of G. If the relative Dehn function is linear, then G is called *hyperbolic relative to* H.

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Theorem(Lee-Sakuma, 2017)

Let $r_0 = [4, 3, 3]$ and $r_i = [4, 2, (i-1)\langle 3 \rangle, 4, 3]$ for every integer $i \geq 1$ be the continued fraction expansions. Also for $n \geq 1$, let

$$G_n = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = u_{r_n} = 1 \rangle,$$

where u_{r_i} is the single relator of the upper presentation of the link group of the 2-bridge link of slope r_i for every integer $i \geq 0$. Then the direct limit of a sequence $G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots$ equipped with the canonical epimorphism $\alpha_n : G_n \twoheadrightarrow G_{n+1}$ at each $n \geq 0$ is $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1 \rangle$ which is non-Hopfian.

- 2-bridge links are classified by rational numbers less than 1.
- We can obtain the relators of 2-bridge link groups in G_n from these continued fractions.

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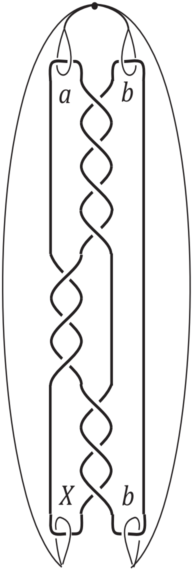

- $G_0 = \langle a, b \,|\, u_{r_0} = 1 \rangle$: the 2-bridge link group of slope $r_0 = [4, 3, 3]$

  $u_{r_0} \equiv \underbrace{ababa}_{5}\underbrace{b^{-1}a^{-1}b^{-1}a^{-1}}_{4}\underbrace{baba}_{4}\cdots.$

- $u_{r_0} \to u_{r_1}$

  $X \equiv abab^{-1}a^{-1}b^{-1}abab^{-1}a^{-1}b^{-1}a^{-1}baba^{-1}b^{-1}a^{-1}$ $(X \rightsquigarrow [4, 3])$

  $f : F(a, b) \to F(a, b)$: the homomorphism defined by $a \mapsto X^{-1}$ and $b \mapsto b^{-1}$

  $\Rightarrow f(u_{r_0}) =_{G_0} u_{r_1}$.

- $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1 \rangle$, where $r_0 = [4, 3, 3]$ and $r_i = [4, 2, (i-1)\langle 3 \rangle, 4, 3]$ for $i \geq 1$.

Theorem(Kim-Lee, Preprint)

Let $r_i = [6, i\langle 2,1,1\rangle, 3, 4]$ for every integer $i \geq 0$. Then for each integer $n \geq 0$, the group $G_n = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = u_{r_n} = 1\rangle$ is hyperbolic relative to a set $\{H_n, K_n\}$ of subgroups. Here, $H_n = \langle a, v_{r_0}, v_{r_1}, \cdots, v_{r_n}\rangle$ and $K_n = \langle b, w_{r_0}, w_{r_1}, \cdots, w_{r_n}\rangle$ are proper subgroups of G_n, where $(u_{r_i}) \equiv (a v_{r_i} a^{-1} v_{r_i}^{-1}) \equiv (w_{r_i} b w_{r_i}^{-1} b^{-1})$ for every $i \geq 0$.

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Let G be the direct limit of a sequence

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equipepd with the cannonical epimorphism $\alpha_n : G_n \twoheadrightarrow G_{n+1}$, where G_n is in the above, is infinitely presented as
$G = \langle a, b \,|\, u_{r_0} = u_{r_1} = u_{r_2} = \cdots = 1\rangle$, is non-Hopfian.

- G is infinitely presented, and we proved that G is non-residually finite by the non-Hopfian property.
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