Recent progress on graphs with fixed smallest eigenvalue

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Outline

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Graph: $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$.

- All graphs in this talk are undirected and simple.
- The adjacency matrix A of a graph Γ is the matrix whose rows and columns are indexed by its vertices such that $A_{xy} = 1$ if xy is an edge and 0 otherwise.

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- In the third part, I will also be interested in the second largest eigenvalue.

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Smallest eigenvalue -2

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We say a connected graph G with smallest eigenvalue at least -2 and adjacency matrix A is a generalized line graph if there exists an integral matrix N such that $A + 2I = N^T N$.

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Note that if I can take N with entries only $0'$s and $1'$s, then G is a line graph. So a generalized line graph is a generalization of a line graph.

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We give now a sketch of proof for this result.

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- The irreducible root lattices were classified by Witt, and are of type A_n, D_n or E_6, E_7, E_8.
- The first two lattices give us generalized line graphs, and for the last three lattices one can show that the number of vertices is at most 36.

Outline

Smallest eigenvalue $-1-\sqrt{2}$

Hoffman (1977) showed the following result:

Theorem

Let $2 \leq \lambda < 1+\sqrt{2}$. Then there is constant $K = K(\lambda)$ such that if Γ is a connected graph with minimal valency at least K and smallest eigenvalue $\lambda_{\text{min}} \geq -\lambda$, then Γ is a generalised line graph. In particular, $\lambda_{\text{min}} \geq -2$.

- This result means that there exists a real number $\tau(k) < -2$ such that any connected graph with minimal valency at least k has smallest eigenvalue either at least -2 or at most $\tau(k)$, and $\tau(k) \rightarrow -1-\sqrt{2}$ $(k \rightarrow \infty)$.

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- Woo and Neumaier (1995) generalized this result by Hoffman by going slightly below $-1-\sqrt{2}$.

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- Note that B has only integral entries.

Rephrasing the results of Cameron et al. and Hoffman

As the generalised line graphs are exactly the graphs with smallest eigenvalue at least -2 which are 1-integrable and and any graph represented by E_8 is 2-integrable, we can rephrase the results of Cameron et al. and Hoffman as follows:

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Can we generalize these two results to graphs with smallest eigenvalue at least -3? In this talk we will show a generalization of the result of Hoffman, and that to generalize the first result is probably difficult.

Outline

Main result

Our main result is:

Theorem

There exists a constant $K > 0$ such that any connected graph G with minimal valency at least K and $\lambda_{\min}$ at least -3 is 2-integrable.

Remarks

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- Adding three vertices to the McLaughlin graph with parameters $(275, 162, 105, 81)$, K., Munemasa, Rehman and Yang showed that $K \geq 166$. We will come back to this later.
- We only know an implicit upper bound for K, but certainly our bound is far from the true value.
- We have a family of connected non 2-integrable graphs with unbounded number of vertices and smallest eigenvalue at least -3 (using the same srg as above). So this means that to generalize the result of Cameron et al. may be difficult.

- The proof is a combination of the techniques used in the result of Cameron et al. of 1976 and the result of Hoffman of 1977.

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Conjecture and Question

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- Can we extend our result to smallest eigenvalue -4?
- Our method can not be extended to -4.
- If this is true then I believe you can easily replace -4 by any negative integer.

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- Let $v(k, \lambda)$ be the maximum order of a connected k-regular graph with second largest eigenvalue at most λ.

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- For $k \geq 3$ integer and all real $\lambda \geq 2\sqrt{k-1}$, it is known that $v(k, \lambda)$ is infinite.

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- For $k \geq 3$ integer and all real $\lambda \geq 2\sqrt{k-1}$, it is known that $v(k, \lambda)$ is infinite.
- Reason: For all integers $k \geq 3$, there exists an infinite family of bipartite k-regular Ramanujan graphs as found by Marcus, Spielman, and Srivastava (2015)

Alon-Boppana Theorem

What about if λ is smaller with respect to k?

Alon-Boppana Theorem

What about if λ is smaller with respect to k? The Alon-Boppana Theorem states:

Theorem

For any integer $k \geq 3$ and real number $\lambda < 2\sqrt{k-1}$, the number $v(k, \lambda)$ is finite.

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- $v(k,\lambda) = k+1$ for $-1 \leq \lambda < 0$ and the unique graph with equality is the complete graph K_{k+1}.

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- Cioabă et al. (2016) showed $v(k, 1) = 2k + 2$ when $k \ge 11$. The equality case is obtained by the complement of the line graph of $K_{2,k+1}$, or, equivalently, $K_{k+1,k+1}$ minus a perfect matching. We denote this graph by $\tilde{K}_{k+1,k+1}$.

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- Also they showed $2k + 2 \leq v(k, 1) \leq 2k + 6$ for all $k \geq 2$.

J.Y. Yang and myself showed the following result when $k >> \lambda$:

Theorem

Let λ be an integer at least 1. Then there exists a constant $C_1(\lambda)$ such that $2k + 2 \leq v(k, \lambda) \leq 2k + C_1(\lambda)$ holds for all $k > \frac{\lambda^2+4}{4}$.

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- Before I discuss the proof I would like to discuss the number $T(\lambda)$.

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- This shows that $T(\lambda) \geq 2\lambda$.
- Conjecture: $T(\lambda) = 2\lambda$.

Outline

Using the theory of Hoffman graphs we were able to show:

Proposition

Let λ be a real number at least 1. Then there exists a constant $M(\lambda) \geq \lambda^3$ such that, if G is a graph satisfying

(i) every pair of vertices at distance 2 has at least $M(\lambda)$ common neighbors,

(ii) the smallest eigenvalue of G, $\lambda_{\min}(G)$, satisfies $\lambda_{\min}(G) \geq -\lambda$,

then G has diameter 2 and any vertex x has at most $\lfloor\lambda\rfloor\lfloor\lambda^2\rfloor$ non-neighbours in G.

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Our main result follows fairly straightforward from this proposition.

More applications

Co-edge-regular graph

A graph G is co-edge-regular with parameters (v, k, c_2), is a k-regular graph with v vertices and with the property that every pair of non-adjacent vertices has exactly c_2 common neighbours.

By applying the proposition and the Alon-Boppana Theorem we obtain:

Theorem

Let λ be a positive real number. Then, there exists a positive integer $C_2(\lambda)$ such for any co-edge regular graph G with parameters (v, k, c_2) and smallest eigenvalue at least $-\lambda$, at least one of the following holds:

- $c_2 \leq C_2(\lambda)$;
- $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$.

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- We can also obtain a similar result for amply regular graphs.

Strongly Regular Graphs

A graph Γ on n vertices is called **strongly regular** (srg) with parameters (n, k, λ, μ) if Γ is k-regular and two distinct vertices have λ resp. μ common neighbours depending whether they are adjacent or not.

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Outline

Geometric SRG

- Recall that if Γ is SRG with smallest ev $\lambda_{\min}$ then a maximal clique has order at most $1 + \frac{k}{-\lambda_{\min}}$ and a clique with this order is called a *Delsarte* clique.

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- A SRG is called *geometric* if the edge set can be partitioned into Delsarte cliques.
- Examples of geometric SRG: $t \times t$-grid, $T(n)$, and so on.
- It is easy to see that a geometric SRG is 1-integrable.
- Also the regular complete multipartite graphs are 1-integrable.

SRG with smallest ev -2

- The square grid graphs, the triangular graphs and the Cocktail Party graphs are all 1-integrable.

SRG with smallest ev -2

- The square grid graphs, the triangular graphs and the Cocktail Party graphs are all 1-integrable.
- All other srg with smallest ev -2 are 2-integrable, but not 1-integrable.

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- First the complement of the Sims-Gewirtz graph, the unique srg *SG* with parameters $(56, 45, 36, 36)$.
- One way to construct *SG* is to take the quasi-symmetric 2-$(21, 6, 4)$-design. Two distinct blocks of this design intersect in either 2 or 0 points.
- Now *SG* has as vertices the blocks and two are adjacent if they intersect in two points. As $\lambda_{\min}(SG) = -3$, it is easily seen that *SG* is 2-integrable using the point-block incidence matrix.

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- It is not so difficult to show that SG is not 1-integrable.

- The complement of the generalized quadrangle $GQ(3,9)$ is the unique strongly regular graph $\overline{GQ}$ with parameters $(112, 81, 60, 54)$.
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- Using the shortened Leech lattice, it can be shown that *McL* is 4-integrable.
- We also showed that the Hoffman-Singleton graph *HS* with parameters $(50, 7, 0, 1)$ is not 2-integrable.
- We do not know whether *HS* is 4-integrable.

Thank you for your attention.