Arithmetics and combinatorics of circulant graphs

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Spanning trees

A *spanning tree* T in a connected graph G a subgraph of G that is a tree and contains all vertices of G.
A spanning tree of a connected graph G can also be defined as a maximal set of edges of G that contains no cycle, or as a minimal set of edges that connect all vertices.
The number of spanning trees is very important invariant of a graph. Sometimes, it also called *complexity* of the graph and denote by $\tau(G)$.
One of the first results on the complexity was obtained by Cayley who proved that $\tau(K_n) = n^{n-2}$, where K_n is the complete graph on n vertices.
The famous Kirchhoff's Matrix Tree Theorem (1847) states that $\tau(G)$ can be expressed as the product of nonzero Laplacian eigenvalues of G divided by the number of its vertices. Since then, a lot of papers devoted to the complexity of various classes of graphs were published. In particular, explicit formulae were derived for complete multipartite graphs, wheels, fans, prisms,ladders, Möbius ladders, lattices, anti-prisms, and for many other families.

Circulant graphs

Circulant graphs can be described in a few equivalent ways:

(a) The graph has an adjacency matrix that is a circulant matrix.

(b) The automorphism group of the graph includes a cyclic subgroup that acts transitively on the graph's vertices.

(c) The graph is a Cayley graph of a cyclic group.

Examples

(a) The circulant graph $C_n(s_1, \ldots, s_k)$ with jumps $s_1, \ldots, s_k$ is defined as the graph with n vertices labeled $0, 1, \ldots, n-1$ where each vertex i is adjacent to $2k$ vertices $i \pm s_1, \ldots, i \pm s_k \mod n$.

(b) n-cycle graph $C_n = C_n(1)$.

(c) n-antiprism graph $C_{2n}(1, 2)$.

(d) n-prism graph $Y_n = C_{2n}(2, n)$, n odd.

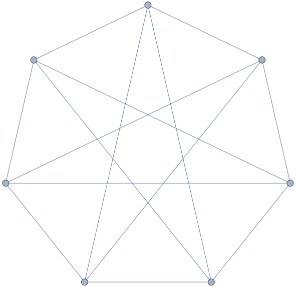
(e) The Moebius ladder graph $M_n = C_{2n}(1, n)$.

(f) The complete graph $K_n = C_n(1, 2, \cdots, [\frac{n}{2}])$.

(g) The complete bipartite graph $K_{n,n} = C_n(1, 3, \cdots, 2[\frac{n}{2}] + 1)$.

Circulant graphs $C_n(1,3)$ for $n = 7$ is shown below

Out[3]=



In this presentation we investigate the infinite family of circulant graphs $C_n(s_1, s_2, \ldots, s_k)$ and also the family of circulant graphs with non-constant jumps $C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$, where $1 \le s_1 < s_2 < \ldots, s_k \le [\frac{\beta n}{2}]$, $1 \le \alpha_1 < \alpha_2 < \ldots, \alpha_\ell \le [\frac{\beta}{2}]$.
We present an explicit formula for the number of spanning trees in both cases. This formula is given in terms of the Chebyshev polynomials of the first kind.

Next, we provide some arithmetic properties of the complexity function. We show that the number of spanning trees of the circulant graph can be represented in the form $\tau(n) = p\, n\, a(n)^2$, where $a(n)$ is an integer sequence and p is a prescribed natural number depending of parity of β and n.

Finally, we find an asymptotic formula for $\tau(n)$ through the Mahler measure of the Laurent polynomials differing by a constant from

$$2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}).$$

Kirchhoff theorem

By the celebrated Kirchhoff theorem, the number of spanning trees $\tau(n)$ is equal to the product of nonzero eigenvalues of the Laplacian of a graph $C_n(s_1, s_2, \ldots, s_k)$ divided by the number of its vertices n. To investigate the spectrum of Laplacian matrix, we denote by $T = circ(0, 1, \ldots, 0)$ the $n \times n$ shift operator. Consider the Laurent polynomial

$$L(z) = 2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}).$$

Then the Laplacian of $C_n(s_1, s_2, \ldots, s_k)$ is given by the matrix

$$\mathbb{L} = L(T) = 2kI_n - \sum_{i=1}^{k}(T^{s_i} + T^{-s_i}).$$

Kirchhoff theorem

The Laplacian matrix $\mathbb{L}$ has eigenvalues $\lambda_j = L(\varepsilon_n^j) = 2k - \sum_{i=1}^{k}(\varepsilon_n^{js_i} + \varepsilon_n^{-js_i})$, where $j = 0, \ldots, n-1$. Since the graph under consideration is connected, we have $\lambda_0 = 0$ and $\lambda_j > 0$, $j = 1, 2, \ldots, n-1$. Hence

$$\tau(n) = \frac{1}{n}\prod_{j=1}^{n-1} L(\varepsilon_n^j).$$

To continue the proof we replace the Laurent polynomial $L(z)$ by a true polynomial $P(z) = (-1)z^{s_k}L(z)$. Then $P(z)$ is a monic polynomial of the degree $2s_k$ with the same roots as $L(z)$. We note that

$$\prod_{j=1}^{n-1} P(\varepsilon_n^j) = (-1)^{(n-1)}\varepsilon_n^{\frac{(n-1)n}{2}s_k}\prod_{j=1}^{n-1} P(\varepsilon_n^j) = (-1)^{(s_k-1)(n-1)}\prod_{j=1}^{n-1} L(\varepsilon_n^j). \tag{1}$$

We can recognize the complex numbers $\varepsilon_n^j,\ j = 1, \ldots, n-1$ as the roots of polynomial $\frac{z^n-1}{z-1}$. Then, by the basic properties of resultant we have

$$\prod_{j=1}^{n-1} P(\varepsilon_n^j) = \mathrm{Res}(P(z), \frac{z^n-1}{z-1}) = \mathrm{Res}(\frac{z^n-1}{z-1}, P(z))$$

$$= \prod_{z:P(z)=0} \frac{z^n-1}{z-1} = n^2 \prod_{j=1}^{s_k-1} \frac{z_j^n-1}{z_j-1}\frac{z_j^{-n}-1}{z_j^{-1}-1} =$$

$$= n^2 \prod_{j=1}^{s_k-1} \frac{2T_n(w_j)-2}{2w_j-2} = n^2 \prod_{j=1}^{s_k-1} \frac{T_n(w_j)-1}{w_j-1}, \tag{2}$$

where $w_j = \frac{1}{2}(z_j + \frac{1}{z_j})$. Combine (1) and (2) we have the following formula for the number of spanning trees

$$\tau(n) = (-1)^{(s_k-1)(n-1)} n \prod_{j=1}^{s_k-1} \frac{T_n(w_j)-1}{w_j-1}. \tag{3}$$

Enumeration of spanning trees

Theorem

The number of spanning trees in the circulant graph $C_n(s_1, s_2, \ldots, s_k)$ is given by the formula

$$\tau(n) = \frac{n}{q} \prod_{p=1}^{s_k-1} |2\, T_n(w_p) - 2|, \tag{4}$$

where $q = s_1^2 + s_2^2 + \ldots + s_k^2$ and w_p, $p = 1, 2, \ldots, s_k - 1$ are different from 1 roots of the equation $\sum_{j=1}^{k} T_{s_j}(w) = k$, and $T_k(w)$ is the Chebyshev polynomial of the first kind.

Enumeration of spanning trees

Theorem

The number of spanning trees in the circulant graphs with non-constant jumps $C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$, *where* $1 \leq s_1 < s_2 < \ldots < s_k \leq [\frac{\beta n}{2}]$, $1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_\ell \leq [\frac{\beta}{2}]$ *is given by the formula*

$$\tau(n) = \frac{n}{\beta\, q} \prod_{p=1}^{\beta} \prod_{\substack{h=1, \\ w_\beta(h) \neq 1}}^{s_k} |2T_n(w_p(h)) - 2\cos(\frac{2\pi p}{\beta})|,$$

where for each $p = 1, 2, \ldots, \beta$ *the numbers* $w_p(h)$, $h = 1, 2, \ldots, s_k$, *are roots of the equation* $\sum_{j=1}^{k} T_{s_j}(w) = k + 2\sum_{q=1}^{\ell} \sin^2(\frac{\pi p \alpha_q}{\beta})$, $T_k(w)$ *is the Chebyshev polynomial of the first kind and* $q = s_1^2 + s_2^2 + \ldots + s_k^2$.

As the first consequence from the above Theorem we have the following result obtained earlier by Justine Louis (2016) in a slightly different form.

Corollary

The number of spanning trees in the circulant graphs with non-constant jumps $C_{\beta n}(1, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$, *where* $1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_\ell \le [\frac{\beta}{2}]$ *is given by the formula*

$$\tau(n) = \frac{n\,2^{\beta-1}}{\beta} \prod_{p=1}^{\beta-1} (T_n(1 + 2\sum_{q=1}^{\ell} \sin^2(\frac{\pi p\,\alpha_q}{\beta})) - \cos(\frac{2\pi p}{\beta})),$$

where $T_n(w)$ *is the Chebyshev polynomial of the first kind.*

The next explicit formula for the number of spanning trees is new.

Corollary

The number of spanning trees in the circulant graphs with non-constant jumps $C_{\beta n}(1, 2, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$, *where* $1 \le \alpha_1 < \alpha_2 < \ldots, \alpha_\ell \le [\frac{\beta}{2}]$ *is given by the formula*

$$\tau(n) = \frac{nF_n^2}{\beta} \prod_{p=1}^{\beta-1} \prod_{j=1}^{2} |2T_n(w_j(p)) - 2\cos(\frac{2\pi p}{\beta})|,$$

where F_n *is the n-th Fibonacci number,* $T_n(w)$ *is the Chebyshev polynomial of the first kind and* $w_{1,2}(p) = (-1 \pm \sqrt{16 + 25\sum_{q=1}^{\ell} \sin^2(\frac{\pi p \alpha_q}{\beta})})/4.$

We note that nF_n^2 is the number of spanning trees in the graph $C_n(1, 2)$.

Arithmetic properties of complexity

Recall that any positive integer p can be uniquely represented in the form $p = q\,r^2$, where p and q are positive integers and q is square-free. We will call q the *square-free part* of p.

Theorem

Let $\tau(n)$ be the number of spanning trees of the circulant graph $C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$, where $1 \le s_1 < s_2 < \ldots < s_k \le [\frac{\beta n}{2}]$, $1 \le \alpha_1 < \alpha_2 < \ldots, \alpha_\ell \le [\frac{\beta}{2}]$. Denote by p and q the number of odd elements in the sequences $s_1, s_2, \ldots, s_k$ and $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ respectively. Let r be the square-free part of p and s be the square-free part of $p + q$. Then there exists an integer sequence $a(n)$ such that

1^0 $\tau(n) = \beta\, n\, a(n)^2$, *if β and n are odd;*

2^0 $\tau(n) = \beta\, r\, n\, a(n)^2$, *if n is even;*

3^0 $\tau(n) = \beta\, s\, n\, a(n)^2$, *if β is even and n is odd.*

Asymptotic for the number of spanning trees

Theorem

The number of spanning trees in the circulant graph

$$C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n),$$

$1 \le s_1 < s_2 < \ldots < s_k \le \ [\frac{\beta n}{2}]$, $1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_\ell \le [\frac{\beta}{2}]$ $\gcd(s_1, s_2, \ldots, s_k) = d$, $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta) = \delta$, *and* d *and* δ *are relatively prime has the following asymptotic*

$$\tau(n) \sim \frac{n\, d^2 \delta^2}{\beta\, q} A^n, \text{ as } n \to \infty,$$

where $q = s_1^2 + s_2^2 + \ldots + s_k^2$, $A = \prod_{u=1}^{\beta} M(L_p)$ *and* $M(L_p) = \exp(\int_0^1 \log|L_p(e^{2\pi i t})|dt)$ *is the Mahler measure of Laurent polynomial* $L_p(z) = 2k - \sum\limits_{i=1}^{k}(z^{s_i} + z^{-s_i}) + 4\sum\limits_{j=1}^{\ell}\sin^2(\frac{\pi\alpha_j p}{\beta})$.

As an immediate consequence of previous Theorem we have the following result.

Corollary

The thermodynamic limit of the sequence $C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$ of circulant graphs is equal to the arithmetic mean of small Mahler measures of Laurent polynomials $L_u,\ u = 1, 2, \ldots, \beta$. More precisely,

$$\lim_{n\to\infty} \frac{\log \tau(C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n))}{\beta\, n} = \frac{1}{\beta}\sum_{p=1}^{\beta} m(L_p),$$

where $m(L_p) = \int\limits_0^1 \log|L_p(e^{2\pi i t})|dt$ *and*

$L_p(z) = 2k - \sum\limits_{i=1}^{k}(z^{s_i} + z^{-s_i}) + 4\sum\limits_{j=1}^{\ell}\sin^2(\frac{\pi\alpha_j p}{\beta}).$

Examples

Graph $C_{2n}(1,n)$. (Möbius ladder with double steps).
$\tau(C(1,n;2n)) = n\,(T_n(3)+1)$.
Recall that the number of spanninig trees in Möbius ladder with single steps is given by the formula $n\,(T_n(2)+1)$.

Graph $C_{2n}(1,2,n)$.
$\tau(C(1,2,n;2n)) = 4nF_n^2|T_n(\frac{-1-\sqrt{41}}{4})-1||T_n(\frac{-1+\sqrt{41}}{4})-1|$.

Graph $C_{2n}(1,2,3,n)$.
$\tau(n) = \frac{8n}{7}(T_n(\theta_1)-1)(T_n(\theta_2)-1)\prod_{p=1}^{3}(T_n(\omega_p)+1)$,
where $\theta_{1,2} = \frac{-3\pm\sqrt{-7}}{4}$ and $\omega_p,\ p=1,2,3$ are roots of the cubic equation $2w^3+w^2-w-3=0$. We have $\tau(n)=6n\,a(n)^2$ is n is odd and $\tau(n)=2n\,a(n)^2$ is n is even. Also, $\tau(n)\sim\frac{n}{28}A^n,\ n\to\infty$, where $A\approx 42.4038$.

Examples

Graph $C_{3n}(1, n)$.

$$\tau(C(1, n; 3n)) = \frac{n}{3}(2\,T_n(\frac{5}{2}) + 1)^2 = \frac{n}{3}((\frac{5+\sqrt{21}}{2})^n + (\frac{5-\sqrt{21}}{2})^n + 1)^2.$$

Also, it was noted in Chinese technical report that $c_n = 2\,T_n(\frac{5}{2}) + 1$ satisfies recursive relation $c_n = 6c_{n-1} - 6c_{n-2} + c_{n-3}$ with initial data $c_1 = 6, c_2 = 24, c_3 = 111$.

Graph $C_{3n}(1, 2, n)$.

$$\tau(C(1, 2, n; 3n)) = \frac{n}{3}F_n^2(2\,T_n(\omega_1) + 1)^2(2\,T_n(\omega_2) + 1)^2,$$

where $\omega_{1,2} = \frac{-1\pm\sqrt{37}}{4}$.

Examples

Graph $C_{6n}(1, n, 3n)$.

$$\tau(C(1, n, 3n; 6n)) = \frac{n}{3}(4\,T_n(\frac{7}{2})^2 - 1)^2(T_n(5) + 1).$$

Graph $C_{12n}(1, 3n, 4n)$.

$$\tau(n) = \frac{2n}{3} T_n(2)^2(2\,T_n(\frac{5}{2}) + 1)^2(T_n(3) + 1)(4\,T_n(\frac{7}{2})^2 - 3)^2(2\,T_n(\frac{9}{2}) - 1)^2.$$