On mixed Moore-Cayley graphs

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G2R2, Novosibirsk State University, 2018

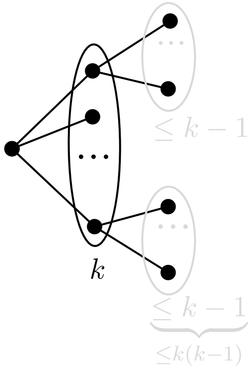
Outline

- Moore bounds, Moore graphs
- (Known) mixed Moore graphs (all are Cayley!)
- (No new) mixed Moore-Cayley graphs (of order ≤ 485)
  (Erskine, 2017)
- Higman's idea (and a Moore graph of valency 57)
- No new mixed Moore-Cayley graphs of some orders

The Moore bound

Let Γ be an undirected graph:

- regular of valency k,
- of diameter D,
- on N vertices.



$$N \leq 1 + k + k(k-1) + \ldots + k(k-1)^{D-1}$$

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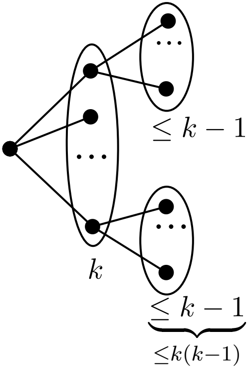

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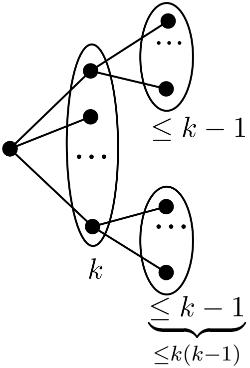

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- regular of valency k,
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- on N vertices.

Then:

$$N \leq 1 + k + k(k-1) + \ldots + k(k-1)^{D-1},$$

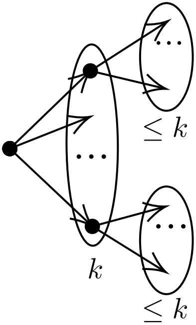
and if equality attains (Damerell, Bannai&Ito):

Diameter D	Valency k	Moore graph	Transitivity
1	k	K_{k+1}	$\surd$
D	2	C_{2D+1}	$\surd$
2	3	Petersen	$\surd$
2	7	Hoffman-Singleton	$\surd$
2	57	?	$\times$

The Moore bound for directed graphs

Let Γ be a directed (only arcs are allowed) graph such that:

- every vertex is of the same out- and in-valency equal to k,
- of diameter D,
- on N vertices.



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The only Moore graphs: a complete graph and a directed cycle.

Plesnik & Znám (1974); Bridges & Toueg (1980).

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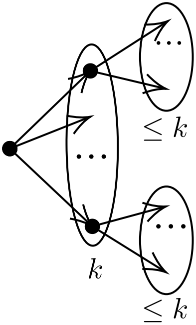

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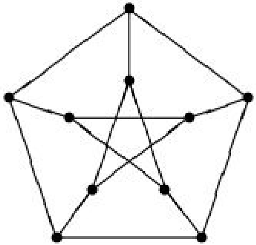

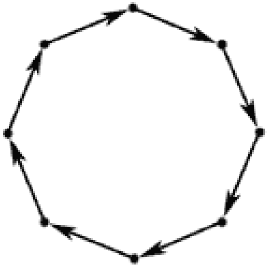

Mixed (partially directed) graphs

Let a graph Γ have arcs as well as (undirected) edges.

The Moore bound can be derived, but it is not pleasant.

We adopt the following property of Moore graphs:
for every ordered pair x, y of vertices of Γ

$$\text{there exists a unique trail } x \rightharpoonup \ldots \rightharpoonup y \qquad (\mathbf{T})$$

of a length not greater than the diameter of Γ.

Theorem (Bosák, 1979)

A graph Γ satisfying ($\mathbf{T}$) is either an undirected tree or a homogeneous graph (a **mixed Moore graph**):

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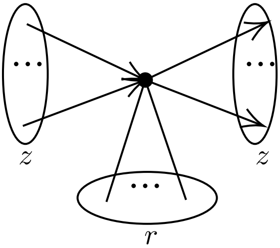

Mixed Moore graphs

Theorem (Bosák, 1979)

Let Γ be a mixed Moore graph of diameter 2 with parameters (r, z). Then the number of vertices of Γ is

$$(z+r)^2 + z + 1$$

and exactly one of the following cases occurs:

- $z = 1$, $r = 0$ (a directed 3-cycle);
- $z = 0$, $r = 2$ (an undirected 5-cycle);
- there exists an odd positive integer c such that

$$\begin{gathered} c \text{ divides } (4z-3)(4z+5), \\ \text{and } r = \tfrac{1}{4}(c^2+3). \end{gathered}$$

Possible values of r: $1, 3, 7, 13, 21, \ldots$,
for given r: infinitely many values of z.

Theorem (Nguyen, Miller, Gimbert, 2007)

There are no mixed Moore graphs with diameter > 2.

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Mixed Moore graphs: the Kautz graphs

The **Kautz graph** $K(z)$ on $(z+2)(z+1)$ vertices is the line graph of a complete directed graph on $z+2$ vertices.

(This is a mixed Moore graph for every z and $r = 1$.)

Theorem (Gimbert, 2001)

The only mixed Moore graphs with $r = 1$ are $K(z)$.

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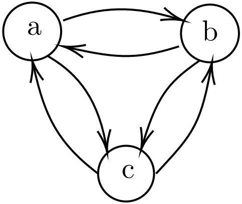

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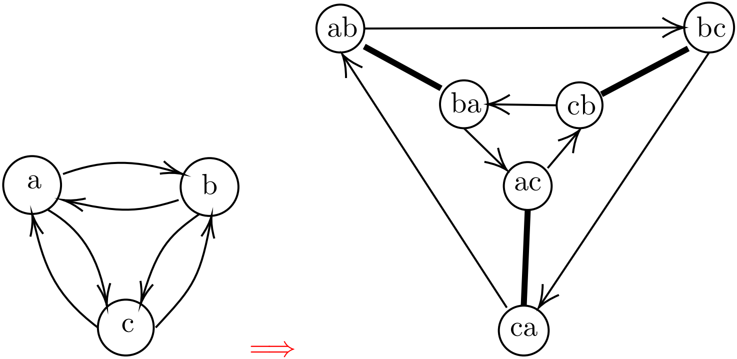

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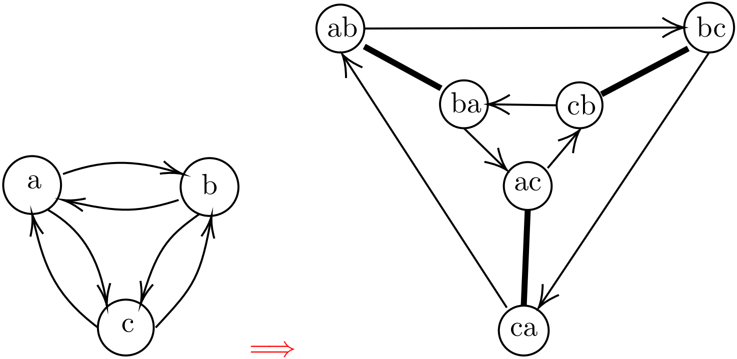

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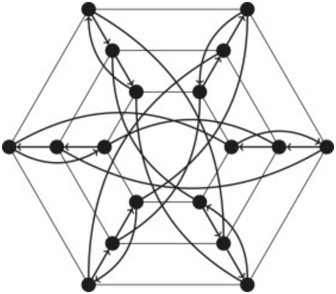
Mixed Moore graphs: the Bosák graph ($r = 3, z = 1$)



Removing all arcs gives a distance regular graph (of diam. 4) known as the Pappus graph.

- $|\mathrm{Aut}(\mathrm{Pappus})| = 216$;
- $|\mathrm{Aut}(\mathrm{Bosák})| = 108$;
- 4 conjugacy classes of regular subgroups of order 18:
  - A Cayley graph of $(S_3 \times S_3) \cap A_6$.

Jørgensen (2015)

Mixed Moore graphs: the Jørgensen graphs $(r = 3, z = 7)$

- A Cayley graph of Aut(Bosák) (108 vertices);
- $|\text{Aut(Jørgensen)}| = 216$;
- Reversing all arcs produces a non-isomorphic (Moore) graph;
- It has two orbits (of length 18 and 108) of subgraphs isomorphic to the Kautz graph on 6 vertices.

Jørgensen (2015)

Mixed Cayley graphs

Given a finite group G and subset $S \subset G \setminus \{1\}$, with $S = S_1 \cup S_2$, $S_1 = S_1^{-1}$ and $S_2 \cap S_2^{-1} = \emptyset$, the **Cayley graph** $Cay(G, S)$ has:

- vertex set G;
- arcs from g to gs for every $g \in G$, $s \in S$;
- the undirected valency $r = |S_1|$;
- the directed valency $z = |S_2|$.

An **automorphism** of Γ is a permutation φ on the vertex set $V(\Gamma)$ that preserves edges and arcs:

$$x \leftrightharpoons y \Longleftrightarrow \varphi(x) \leftrightharpoons \varphi(y).$$

Theorem (Sabidussi)

A graph Γ is a Cayley graph of a group G if and only if G is a **regular** subgroup of $\mathrm{Aut}(\Gamma)$.

Regular action means that for every $x, y \in V(\Gamma)$ there is a unique $\varphi \in G$ such that $\varphi(x) = y$. Equivalently, the only element of G that fixes a vertex is the identity.

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Moore-Cayley graphs

Observation

The only undirected Moore-Cayley graph of diameter 2 is C_5.

(All other Moore graphs have even number of vertices, consider then an involution of G.)

Theorem

The Kautz graphs $K(z)$ are Cayley $\Leftrightarrow z + 2$ is a prime power.

(Regular action on $K(z) \Rightarrow$ sharply 2-transitive action on a complete graph on $z + 2$ vertices $\Rightarrow$ Zassenhaus' theorem)

$Cay(G, S)$ is a mixed Moore graph if and only if:

- for $g \in S$, $\nexists$ pair $(s_1, s_2) \in S \times S$ such that $g = s_1 s_2$;
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Let Γ be a mixed Moore graph of diameter 2. Then:

- Γ contains no undirected cycle of length 3 or 4;
- every arc of Γ is contained in exactly one directed 3-cycle.

Proposition (Erskine, 2017)

Let $\Gamma = Cay(G, S)$ with $S = S_1 \cup S_2$, $S_1 = S_1^{-1}$ and $S_2 \cap S_2^{-1} = \emptyset$ be a mixed Moore graph. Then:

- No element of S_1 has order 3 or 4;
- No pair of elements in S_1 has a product of order 2;
- No two distinct elements of S commute, apart from the inverse pairs in S_1;
- S is product-free, i.e., $S \cap SS = \emptyset$;
- The elements of S_2 are of two types:
  - Elements of order 3;
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Search for more mixed Moore(-Cayley) graphs

n	r	z	Exist	Transitive	Cayley
18	3	1	!		Yes
40	3	3	No[1]		
54	3	4	No[1]		
84	7	2	No[1]		
88	3	6	?	?	No[2]
108	3	7	≥ 2		Yes
150	7	5	?	?	No[2]
154	3	9	?	?	No[2]
180	3	10	?	?	No[2]

[1]: López, Miret, Fernández: *Non-existence of some mixed Moore graphs of diameter* 2 *using SAT* (2016).

[2]: Erskine: *Mixed Moore Cayley graphs* (2017).

Search for more mixed Moore(-Cayley) graphs

n	r	z	Exist	Transitive	Cayley
204	7	7	?	?	No[2]
238	3	12	?	?	No[2]
270	3	13	?	?	No[2]
294	13	4	?	?	No[2]
300	7	10	?	?	No[2]
340	3	15	?	?	No[2]
368	13	6	?	?	No[2]
374	7	12	?	?	No[2]
378	3	16	?	?	No[2]
460	3	18	?	?	No[2]
486	21	1	?	No[3]	

[3]: Jørgensen: talk in Pilsen (2018).

Association scheme

Adjacency matrices of a (symmetric) association scheme $\mathcal{S}$:

$$\mathsf{A}_0 = \mathsf{I}, \mathsf{A}_1, \ldots, \mathsf{A}_D.$$

Their maximal common eigenspaces:

$$\mathbb{R}^X = \mathbf{W}_0 \oplus \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_D$$

	$\mathbf{W}_0$	$\mathbf{W}_1$	$\ldots$	$\mathbf{W}_D$
A_0	P_{00}	P_{01}	$\ldots$	P_{0D}
A_1	P_{10}	P_{11}	$\ldots$	P_{1D}
$\ldots$	$\ldots$	$\ldots$	P_{ij}	$\ldots$
A_D	P_{D0}	P_{D1}	$\ldots$	P_{DD}

Orthogonal projections onto eigenspaces:

$$\mathsf{E}_j : \mathbb{R}^X \mapsto \mathbf{W}_j$$

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$$\langle \mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_D \rangle = \langle \mathsf{E}_0, \mathsf{E}_1, \ldots, \mathsf{E}_D \rangle$$

$$\mathsf{E}_j = \frac{1}{|V|} \sum_{i=0}^{D} Q_{ji} \mathsf{A}_i$$

	A_0	A_1	$\ldots$	A_D
E_0	Q_{00}	Q_{01}	$\ldots$	Q_{0D}
E_1	Q_{10}	Q_{11}	$\ldots$	Q_{1D}
$\ldots$	$\ldots$	$\ldots$	Q_{ij}	$\ldots$
E_D	Q_{D0}	Q_{D1}	$\ldots$	Q_{DD}

Association scheme

$$\langle \mathsf{A}_0, \mathsf{A}_1, \dots, \mathsf{A}_D \rangle = \langle \mathsf{E}_0, \mathsf{E}_1, \dots, \mathsf{E}_D \rangle$$

$$\mathsf{E}_j = \frac{1}{|V|} \sum_{i=0}^{D} Q_{ji} \mathsf{A}_i$$

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Automorphisms

$G \leq \mathrm{Aut}(\mathcal{S})$: permutations on V that preserve the relations
$g \in G \to$ a permutation matrix $\mathsf{X}_g \in \mathbb{R}^{V \times V}$:

$$\mathsf{X}_g^T = \mathsf{X}_g^{-1}, \text{ and } \mathsf{X}_g^n = \mathsf{I} \text{ with } n = |g|, \text{ the order of } g,$$

$$\mathsf{X}_g \mathsf{A}_i \mathsf{X}_g^{-1} = \mathsf{A}_i, \text{ i.e., } \mathsf{X}_g \mathsf{A}_i = \mathsf{A}_i \mathsf{X}_g \text{ for all } i = 0, \ldots, D$$

Every eigenspace $\mathbf{W}_j$ is G-invariant:

$$\mathsf{A}_i(\mathsf{X}_g \overline{w}) = \mathsf{X}_g(\mathsf{A}_i \overline{w}) = P_{ij} \cdot \mathsf{X}_g \overline{w} \text{ for } \overline{w} \in \mathbf{W}_j,$$

in particular:

$$\mathsf{X}_g \mathsf{E}_j \mathsf{X}_g^{-1} = \frac{1}{|V|} \sum_{i=0}^{D} Q_{ji} \underbrace{(\mathsf{X}_g \mathsf{A}_i \mathsf{X}_g^{-1})}_{\mathsf{A}_i} = \mathsf{E}_j$$

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Higman's observation

E_j is a projection matrix $\Rightarrow \mathsf{E}_j^2 = \mathsf{E}_j$, and so the eigenvalues of E_j are only 1's and 0's. Now:

$$\begin{aligned}
(\mathsf{X}_g\mathsf{E}_j)^{|g|} &= (\mathsf{X}_g)^{|g|}(\mathsf{E}_j)^{|g|} && \text{(by } \mathsf{X}_g\mathsf{E}_j = \mathsf{E}_j\mathsf{X}_g\text{)}\\
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is an algebraic integer, while

$$\mathrm{Trace}(\mathsf{X}_g \mathsf{A}_i) = \#\{v \in V \mid (\mathsf{A}_i)_{(v,v^g)} = 1\}$$

In particular, if all the eigenvalues P_{ij} are integers:

- all Q_{ji} are rational by $\mathsf{PQ} = |V|\mathsf{I}$,
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A Moore graph of valency 57

Theorem (G. Higman, unpublished)

A Moore graph $\mathcal{M}$ of valency 57 is not vertex-transitive.

see P.J. Cameron, Permutation Groups (1999).

Suppose not: the number of vertices is 3250, so $\mathcal{M}$ admits an involution g, and let $\mathrm{Fix}(g)$ be the set of its fixed points.

Step 1. $\mathrm{Fix}(g)$ induces either a star or a Moore subgraph.

Step 2. If $x \sim x^g$ for some vertex x then $|\mathrm{Fix}(g)| = 56$.

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Step 5. Suppose g fixes 56 vertices $\Rightarrow |\mathrm{Stab}_G(x)|$ is even $\Rightarrow |G|$ is divisible by 4. Let H denote $G \cap Alt_{3250}$.

Then $|H|$ is even and so $g \in H \subseteq Alt_{3250}$. But g has 56 fixed points and $\frac{3250-56}{2} = 1597$ transpositions.

Theorem (Mačaj, Širáň, 2010; cf. Makhnev, Paduchikh (2001,2007))

The order of $\mathrm{Aut}(\mathcal{M})$ is at most 375.

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$\mathcal{M}$ is strongly regular $\Rightarrow$ an association scheme with 2 classes defined by distances between vertices:

$$\mathsf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 57 & -8 & 7 \\ 3192 & 7 & -8 \end{pmatrix}, \ \mathsf{Q} = \begin{pmatrix} 1 & 1 & 1 \\ \boxed{1520 \quad -\frac{640}{3} \quad \frac{10}{3}} \\ 1729 & \frac{637}{3} & -\frac{13}{3} \end{pmatrix},$$

$$\mathrm{Trace}(\mathsf{X}_g \mathsf{E}_{\mathbf{1}}) = \frac{1}{|V|} \sum_{i=0}^{D} Q_{\mathbf{1}i} \mathrm{Trace}(\mathsf{X}_g \mathsf{A}_i) =$$

$$(\text{recall: } \mathrm{Trace}(\mathsf{X}_g \mathsf{A}_i) = \#\{v \in V \mid (\mathsf{A}_i)_{(v,v^g)} = 1\})$$

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A Moore graph $\mathcal{M}$ of valency 57

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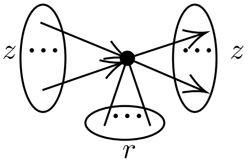
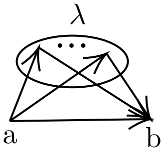
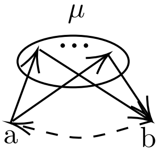
Directed (mixed) strongly regular graphs (v, k, r, λ, μ)

- the total vertex valency is $k = r + z$;
  - every vertex is incident to r edges;
  - every vertex is incident to z in-arcs and z out-arcs;



- for every arc/edge $a \rightharpoonup b$ there exist λ vertices c such that $a \rightharpoonup c \rightharpoonup b$;
- for every non-arc $a \not\longrightarrow b$ there exist μ vertices c such that $a \rightharpoonup c \rightharpoonup b$;





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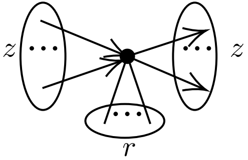

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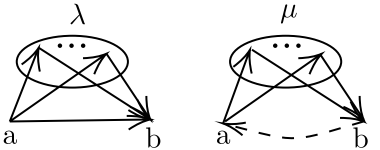

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Adjacency algebra of DSRG

The adjacency matrix A defined by $(\mathsf{A})_{a,b} = 1$ if $a \rightharpoonup b$:

$$\mathsf{A}^2 = r\mathsf{I} + \lambda\mathsf{A} + \mu(\mathsf{J} - \mathsf{A} - \mathsf{I})$$
$$\mathsf{AJ} = \mathsf{JA} = k\mathsf{J},$$

so A is diagonalizable with 3 eigenspaces with eigenvalues k, ρ, σ (can be expressed in terms of v, k, r, λ, μ):

$$\mathsf{A} = k\mathsf{E}_k + \rho\mathsf{E}_\rho + \sigma\mathsf{E}_\sigma,$$
$$\mathsf{I} = \mathsf{E}_k + \mathsf{E}_\rho + \mathsf{E}_\sigma,$$
$$\overline{\mathsf{A}} = (v - k - 1)\mathsf{E}_k - (\rho + 1)\mathsf{E}_\rho - (\sigma + 1)\mathsf{E}_\sigma,$$

where $\overline{\mathsf{A}} = \mathsf{J} - \mathsf{A} - \mathsf{I}$, and E_θ is the projection matrix onto the corresponding (right) θ-eigenspace.

Duval (1988); Jørgensen (2003); Godsil, Hobart, Martin (2007)

In particular,

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Higman's observation applied to DSRG

- $G \leq \mathrm{Aut}(\Gamma)$, $g \to \mathsf{X}_g$;
- $\mathsf{X}_g\mathsf{A} = \mathsf{A}\mathsf{X}_g$;
- $\mathsf{X}_g\mathsf{E}_\theta = \mathsf{E}_\theta\mathsf{X}_g$ for $\theta \in \{k, r, s\}$;
- $\mathsf{E}_\theta^2 = \mathsf{E}_\theta$;
- (E_ρ, E_σ are not symmetric, but we don't need it.)
- Trace($\mathsf{X}_g\mathsf{E}_\theta$) is an (~~algebraic~~) integer;
  - eigenvalues are always integral (Duval, 1988);
- $\mathsf{E}_k, \mathsf{E}_\rho, \mathsf{E}_\sigma \in \mathrm{span}\{\mathsf{A}, \mathsf{J}, \mathsf{I}\}$.

$$\mathrm{Trace}(\mathsf{X}_g\mathsf{E}_\theta) = c_0 \cdot \mathrm{Trace}(\mathsf{X}_g\mathsf{I}) + c_1 \cdot \mathrm{Trace}(\mathsf{X}_g\mathsf{A}) + c_2 \cdot \mathrm{Trace}(\mathsf{X}_g\overline{\mathsf{A}}).$$

Recall:

$$\mathrm{Trace}(\mathsf{X}_g\mathsf{A}) = \#\{v \in V \mid \mathsf{A}_{(v,v^g)} = 1\}$$

Example: a mixed Moore-Cayley graph of order 88

Suppose that a DSRG$(88, 9, 3, 0, 1)$ is a Cayley graph Γ.
Then $\mathrm{Aut}(\Gamma)$ has a regular element g with $|g| = 11$.
So, $\mathrm{Trace}(\mathsf{X}_g \mathsf{I}) = 0$ and by Higman's observation

$$\mathrm{Trace}(\mathsf{X}_g \mathsf{A}) \in \{11, 44, 77\},$$

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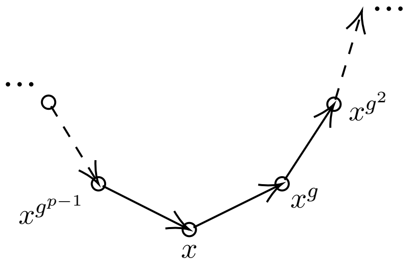

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- g^i has order 11 for $i = 1, \ldots, 10$;
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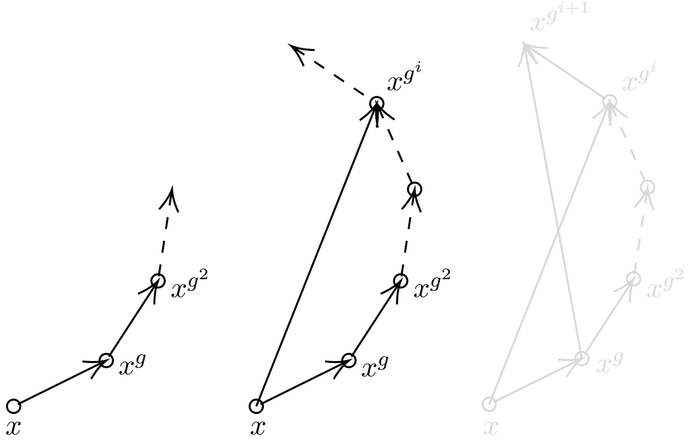
- Γ has 88 vertices $< 10 \times 11 \Rightarrow$



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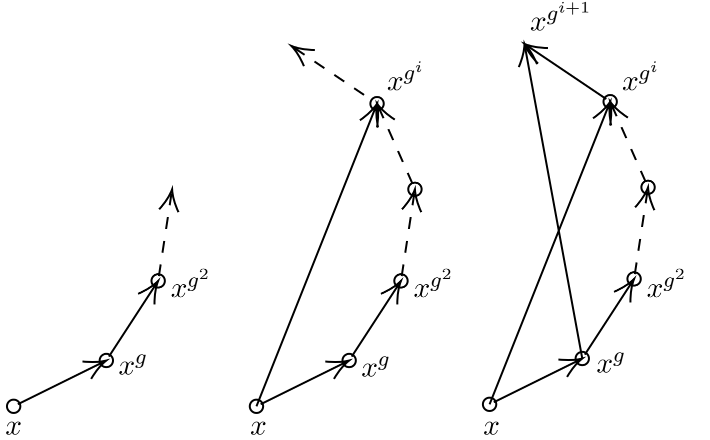

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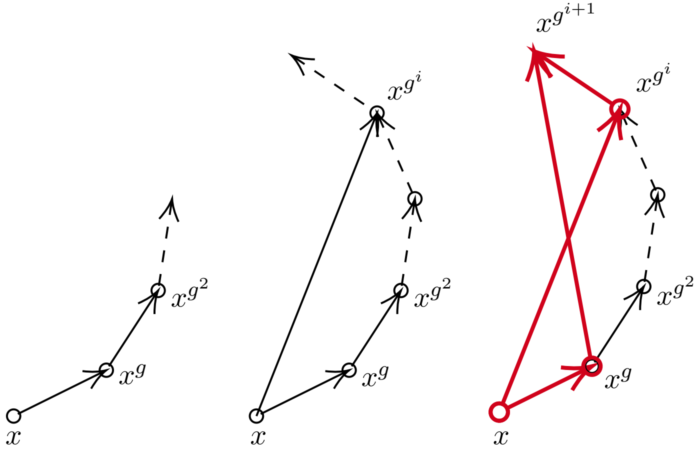

Results

n	r	z	Cayley (by Erskine)	Cayley (by Higman)
40	3	3	No	No
54	3	4	No	
84	7	2	No	No
88	3	6	No	No
150	7	5	No	
154	3	9	No	No
180	3	10	No	

Results

n	r	z	Cayley (by Erskine)	Cayley (by Higman)
204	7	7	No	No
238	3	12	No	No
270	3	13	No	
294	13	4	No	
300	7	10	No	
340	3	15	No	No
368	13	6	No	No
374	7	12	No	No
378	3	16	No	
460	3	18	No	No
486	21	1		

It rules out 29 out of 58 feasible parameter sets for $v \leq 2000$.

Discussion

- More applications of Higman's idea:
  - Temmermans, Thas, Van Maldeghem: *On collineations and dualities of finite generalized polygons* (2009).
  - Makhnev: *On automorphisms of distance-regular graphs* (2010).
  - Mačaj, Širáň: *Search for properties of the missing Moore graph* (2010).
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- We need to know prime factors of $v = (z + r)^2 + z + 1$;
  - (and more from group theory)
- Uniqueness of Jørgensens graphs?
  - Known to be unique as Cayley graphs;
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Köszönöm!

Thank you!

Спасибо!

ありがとうございました!

고맙습니다!

谢谢!

Dank je!

Gracias!

Ďakujem!

Hvala vam!

Grazie!

תודה

Děkuji!

متشكرم

شكرا