Graphs and metrics: the partition case

ISMAEL GONZALEZ YERO

Department of Mathematics, University of Cádiz, Algeciras, Spain
ismael.gonzalez@uca.es

Graphs and Groups, Representations and Relations

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Outline

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1. Introduction
2. Metric generators (and strong ones)
3. Partitions like generators
4. Results on resolving partitions
5. Results on strong resolving partitions

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Navigation of robots in networks

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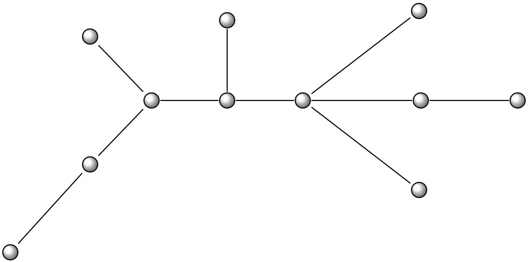

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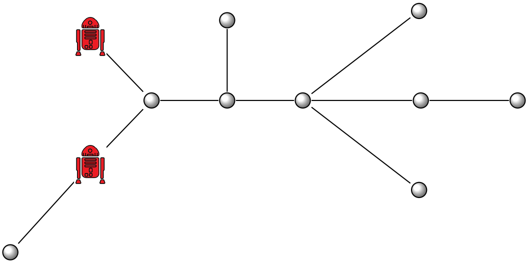

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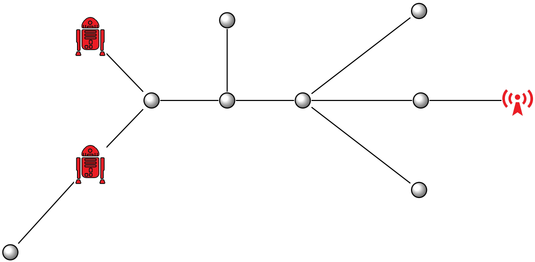

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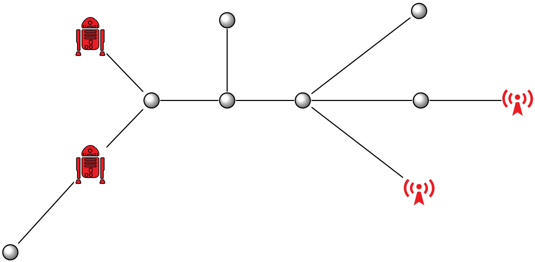

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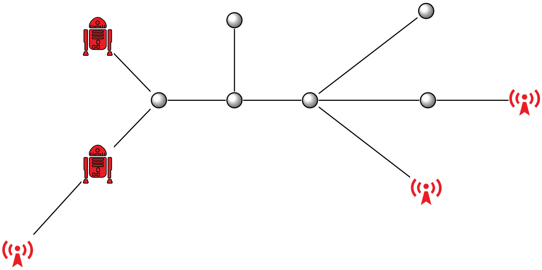

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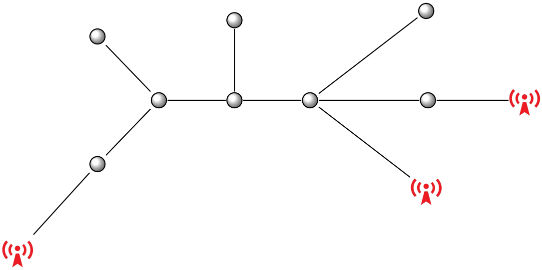

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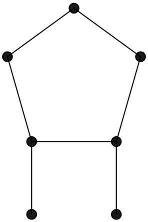

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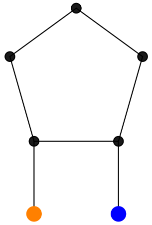

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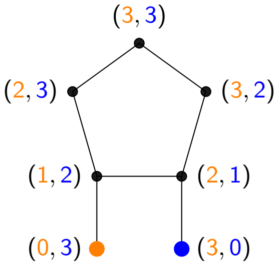

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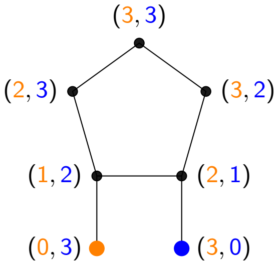

$dim(G)$, *Metric dimension* of G: minimum cardinality of any metric generator.

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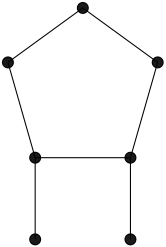

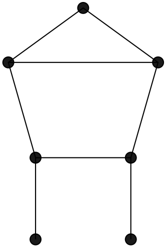

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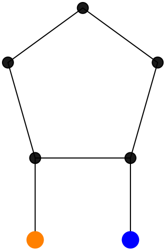

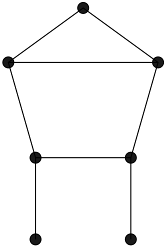

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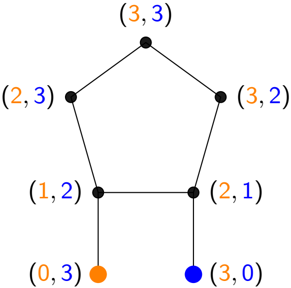

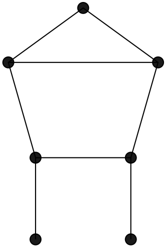

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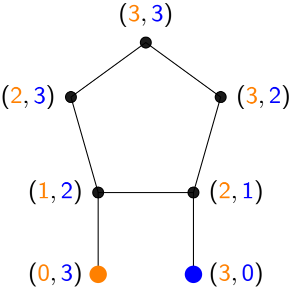

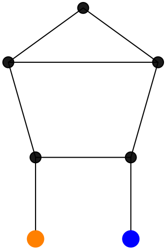

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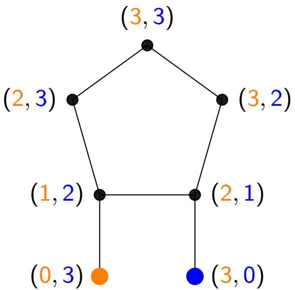

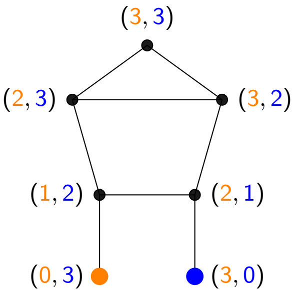

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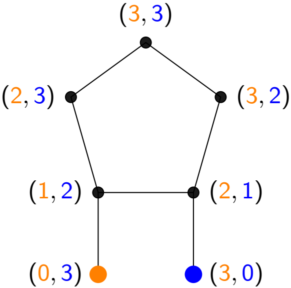

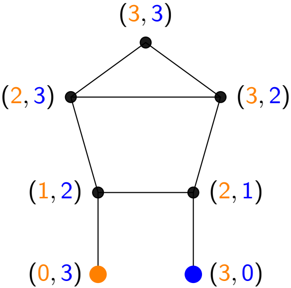

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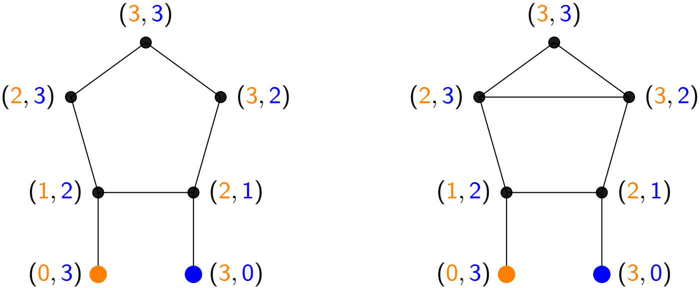

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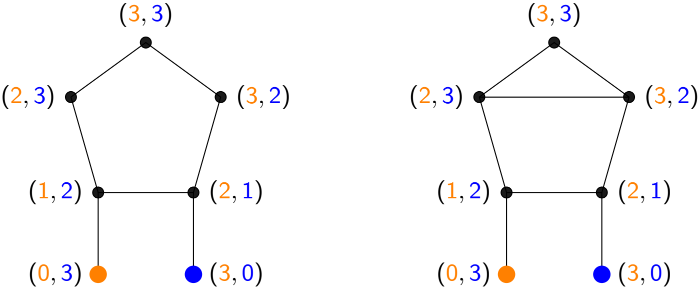

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Does not distinguish graphs.

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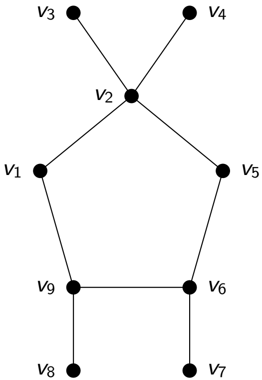

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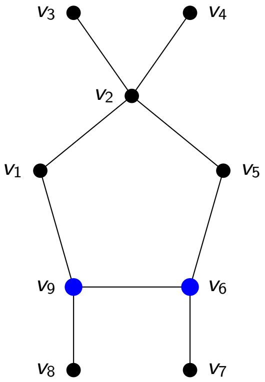

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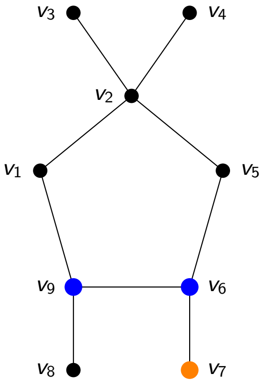

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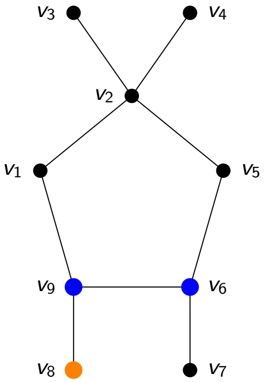

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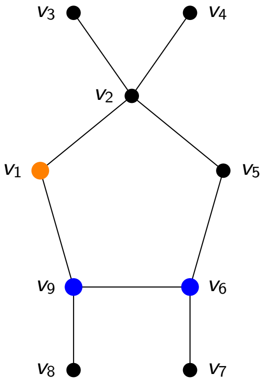

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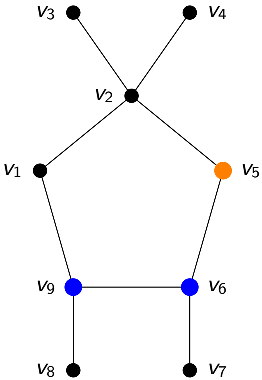

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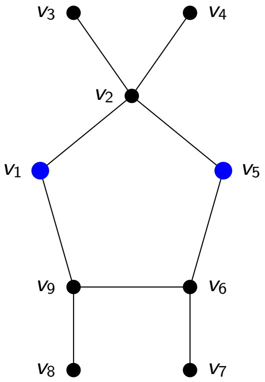

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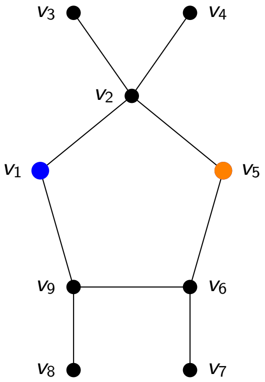

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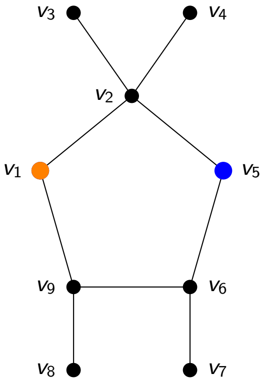

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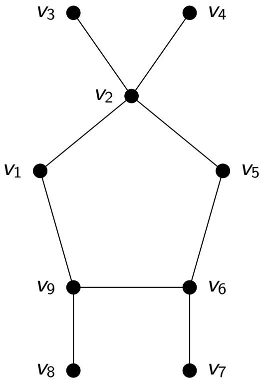

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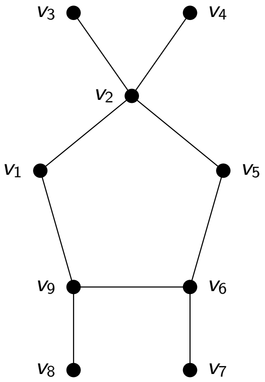

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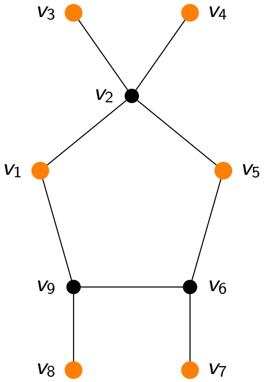

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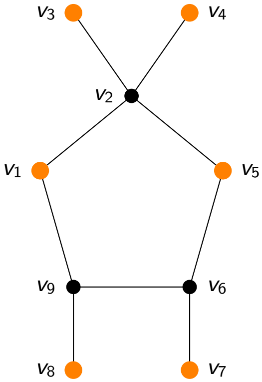

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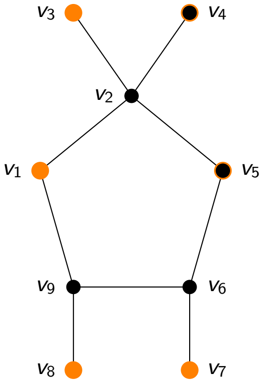

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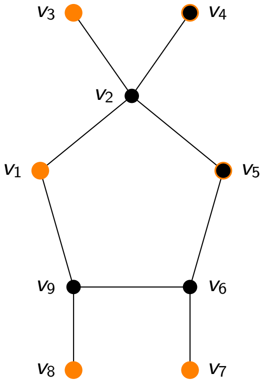

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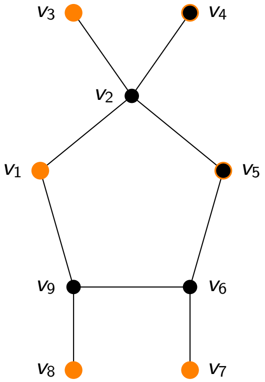

$dim_s(G) = 4$

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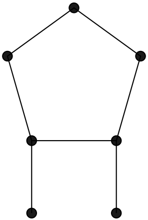

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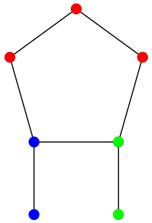

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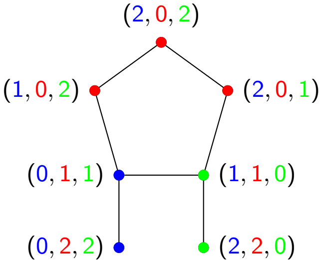

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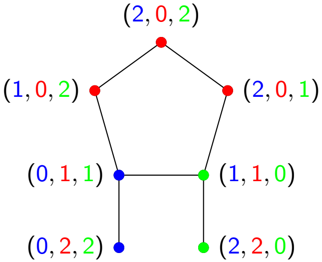

$pd(G)$, *partition dimension* of G: minimum cardinality of any resolving partition.

Strong resolving partitions

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(IGYERO 2014)

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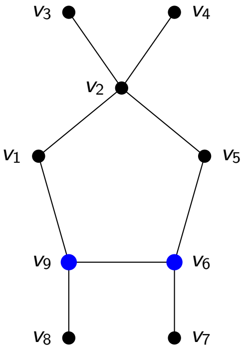

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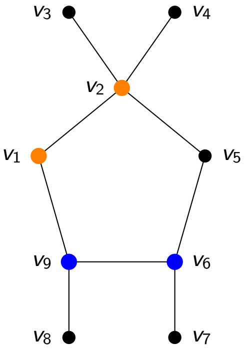

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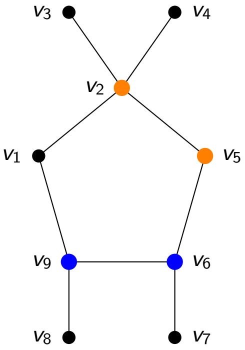

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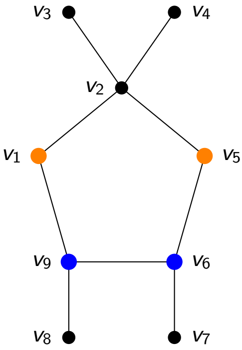

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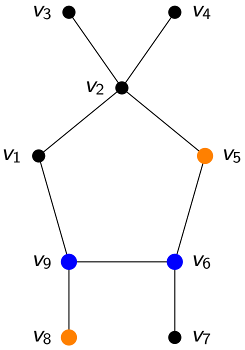

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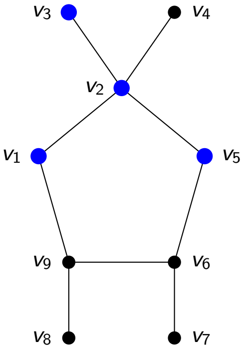

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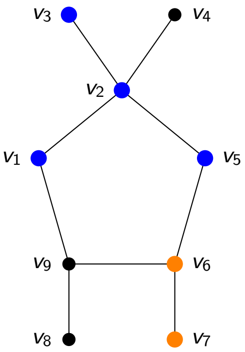

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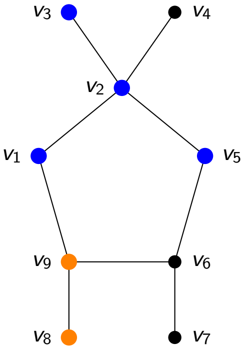

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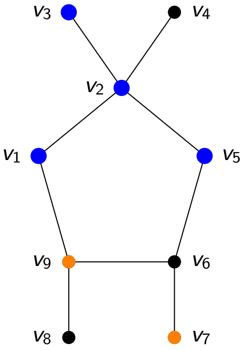

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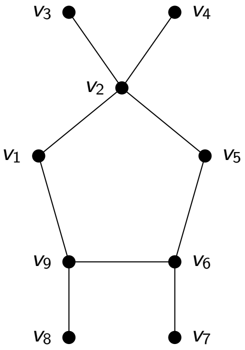

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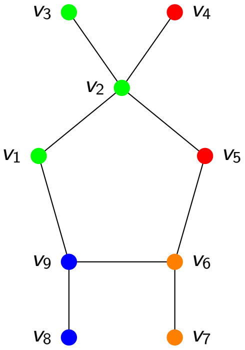

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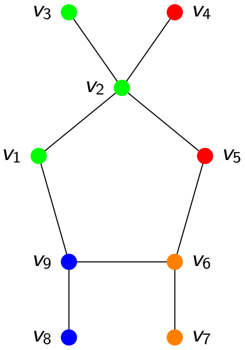

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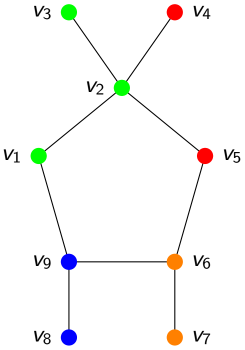

$pd_s(G) \leq 4$

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The first basic relationships

For any connected graph $G = (V, E)$,

$$pd(G) \leq dim(G) + 1 \ \text{ and } \ pd_s(G) \leq dim_s(G) + 1.$$

Outline

[Chartrand, 2000]

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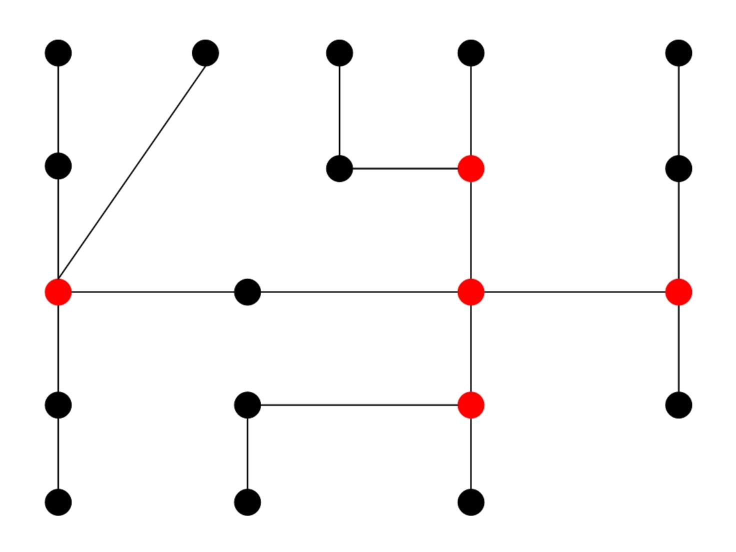

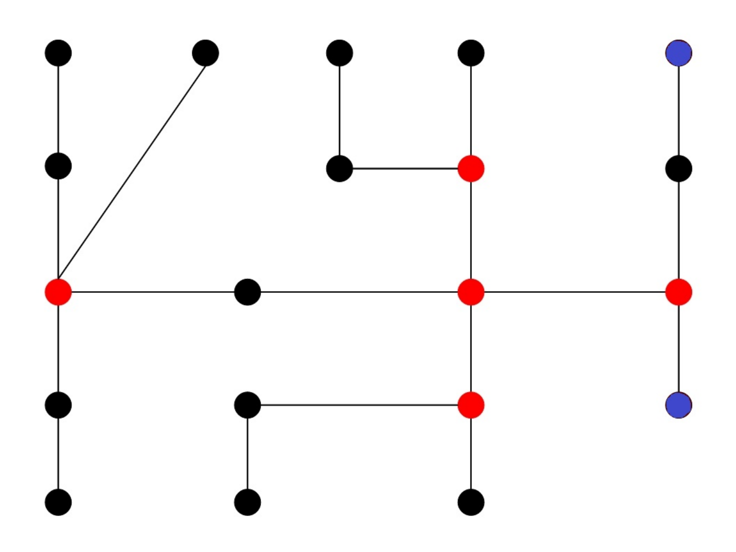

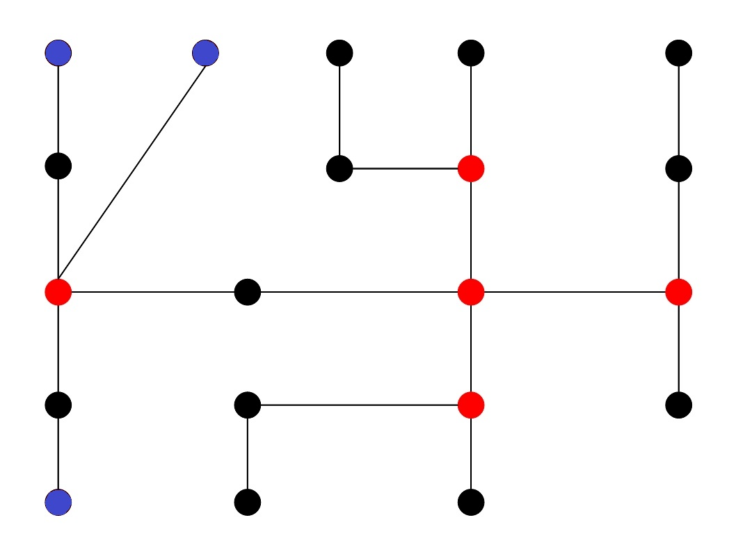

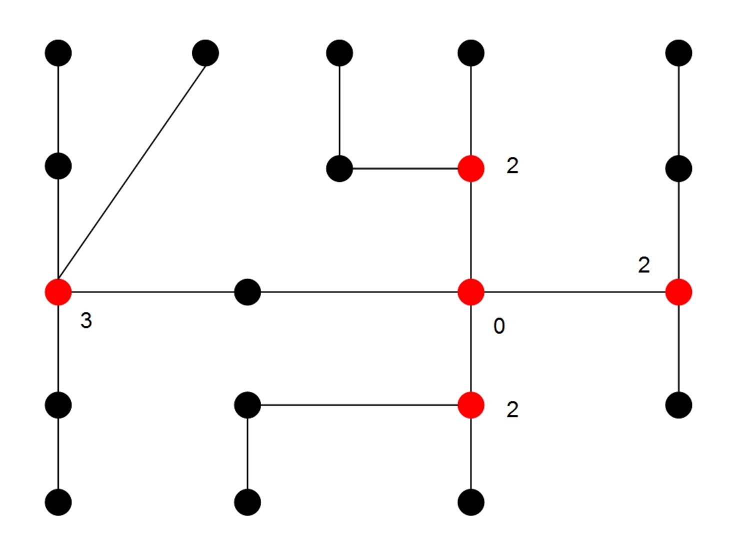
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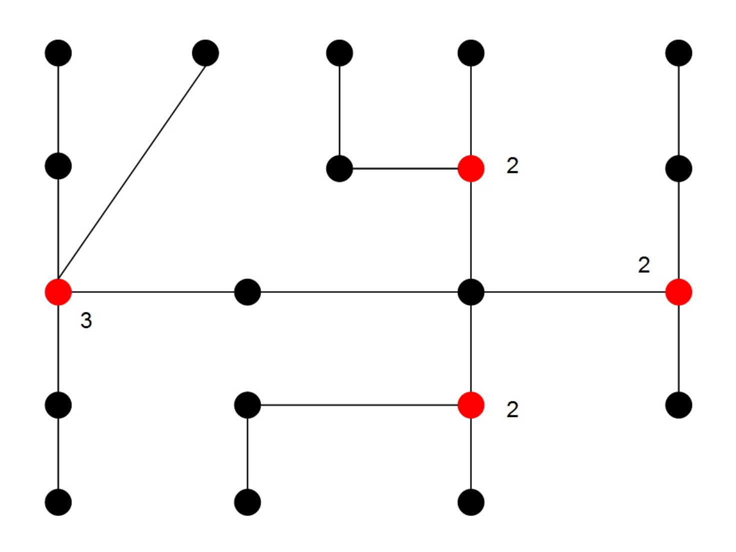
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Metric dimension of trees

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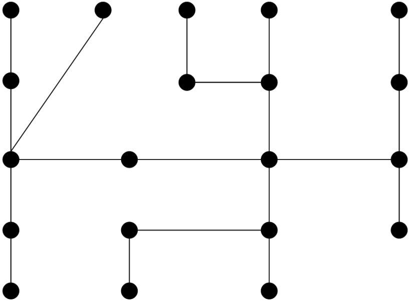

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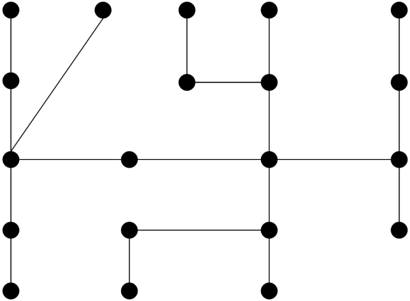

$n_1(T) = 9$ and $ex(T) = 4$

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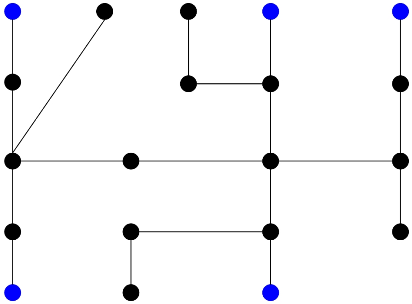

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A conjecture

Finding the partition dimension of a tree is an NP-hard problem.

Cartesian product of two graphs G and H

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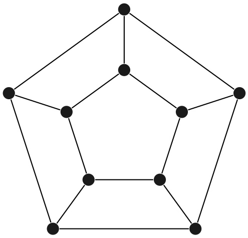

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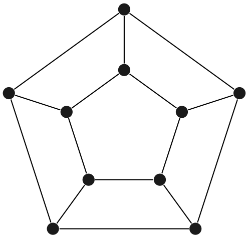

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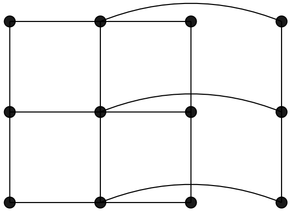

Results, [Pelayo *et al.* - 2007]

For any connected graphs G and H of order n_1 and n_2, respectively.

$$dim(G\Box H) \geq \max\{dim(G), dim(H)\},$$

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- $dim(G\square K_n) \leq dim(G) + n - 2$ if $n \geq 3.$
- $dim(Q_n) \leq n.$
- $dim(P_m\square P_n) = 2.$

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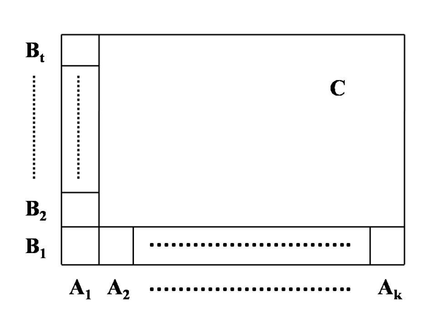

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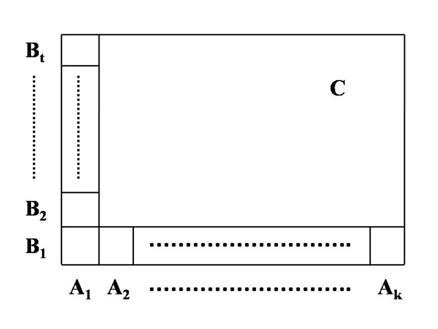

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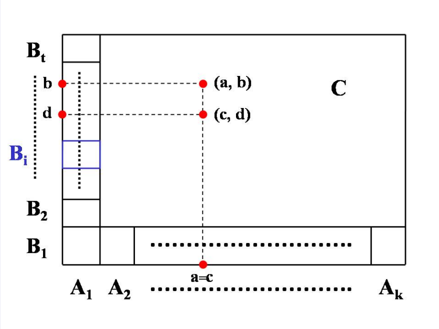

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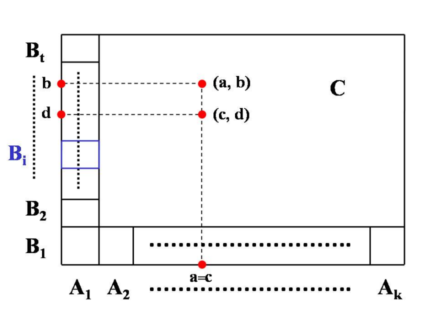

$(a, b), (c, d) \in V_1 \times V_2$.

If $a = c$, then there exist $B_i \in \Pi_2$ such that $d_H(b, B_i) \neq d_H(d, B_i)$ and we use $A_1 \times B_i$.

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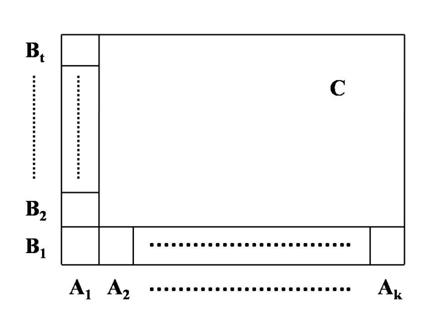

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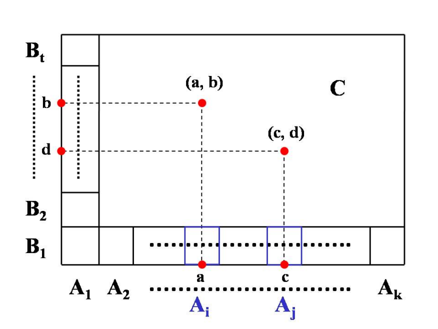

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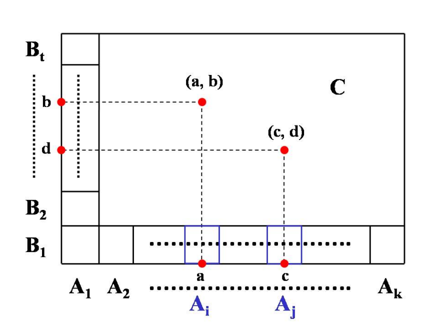

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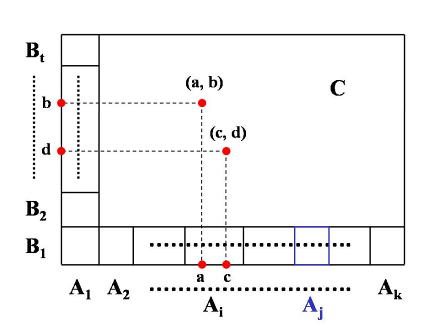

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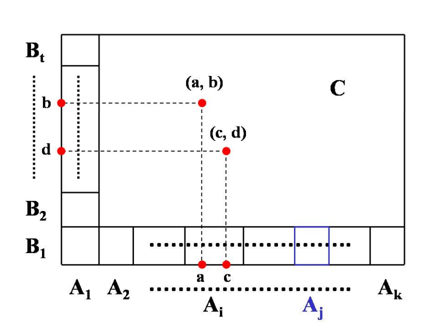

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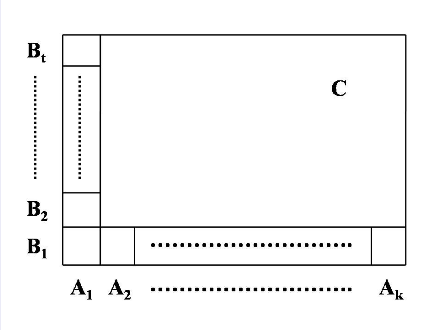

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Then, $r((a, b)|\Pi) \neq r((c, d)|\Pi)$.

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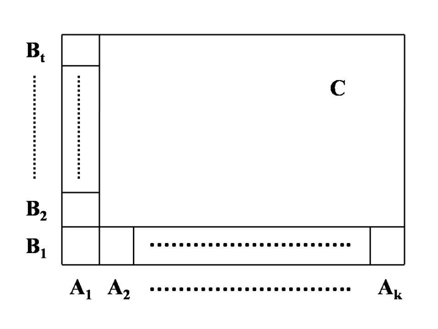

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Π is a resolving partition of $G\Box H$.

WE HAVE NOT BEEN ABLE TO FIND ANY LOWER BOUND!!!

Outline

Some exact values

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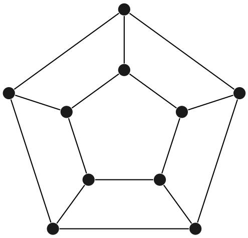
- For any fan graph $F_{1,r}$, $r \geq 3$, $pd_s(F_{1,r}) = \begin{cases} 3, & \text{if } r = 3, 4, \\ \left\lceil \frac{r}{2} \right\rceil, & \text{if } r \geq 5. \end{cases}$

Cartesian product of two graphs G and H

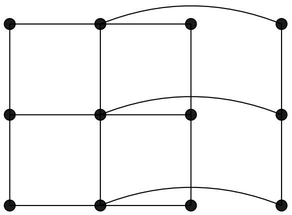
Graph $G\Box H$: $\quad V(G\Box H) = V(G) \times V(H)$

$$(a,b) \sim (c,d) \text{ in } G\Box H \Leftrightarrow \begin{cases} a = c \text{ and } bd \in E(H), & \text{or} \\ ac \in E(G) \text{ and } b = d. \end{cases}$$

$C_5 \Box K_2$:



$K_{1,3} \Box P_3$:



Bounding $pd_s(G\Box H)$

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For any connected graphs G and H,

$$\min\{\omega(G_{SR}), \omega(H_{SR})\} \leq pd_s(G\Box H) \leq \min\{\alpha(G_{SR})|\partial(H)|, |\partial(G)|\alpha(H_{SR})\} + 1.$$

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For any connected graphs G and H or orders n_1 and n_2, respectively,

$$pd_s(G\Box H) \leq \min\{dim_s(G)+n_2, dim_s(H)+n_1\}.$$

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Are the upper bounds above comparable?

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Are the upper bounds above comparable?

	First upper bound	Second upper bound
$P_m\Box P_n$	3	$m+1$
$C_m\Box P_n$	$m+1$	$\min\{m+1, n+\lceil m/2\rceil\}$
$K_m\Box P_n$	$m+1$	$m+1$
$C_m\Box C_n$ $(m \le n)$	$m\lceil n/2\rceil + 1$	$m+\lceil n/2\rceil$
$C_m\Box K_n$	$n\lceil m/2\rceil + 1$	$n+\lceil m/2\rceil$
$K_m\Box K_n$ $(m \le n)$	$n(m-1)+1$	$m+n-1$

Bounding $pd_s(G \square H)$ for some specific cases

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For any integers $m \geq 4$ and $n \geq 2$,

$$pd_s(C_m \square P_n) \leq 4.$$

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For any integers $m \geq 4$ and $n \geq 2$,

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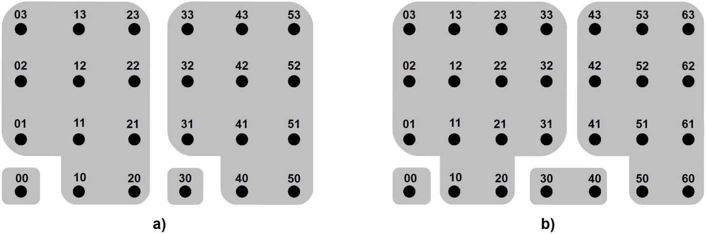

Figure: The partition Π of $C_6 \boxtimes P_4$ and $C_7 \boxtimes P_4$. Vertex labeled by ij represents the vertex (u_i, v_j). Edges of the graphs have not been drawn.

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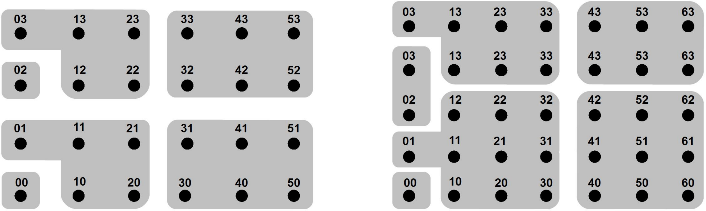

a) b)

Figure: The partition Π of $C_6 \boxtimes C_4$ and $C_7 \boxtimes C_5$. Vertex labeled by ij represents the vertex (u_i, v_j). Edges of the graphs have not been drawn.

An equality and a conjecture

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For any grid graph $P_m \Box P_n$ with $m, n \geq 2$, $pd_s(P_m \Box P_n) = 3$.

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A conjecture

For any integers $m, n \geq 4$,

$$pd_s(C_m \Box P_n) = 4$$

and

$$pd_s(C_m \Box C_n) = 6.$$

THANKS !