On directed strongly regular graphs

Štefan Gyürki

Matej Bel University, Banská Bystrica, Slovakia

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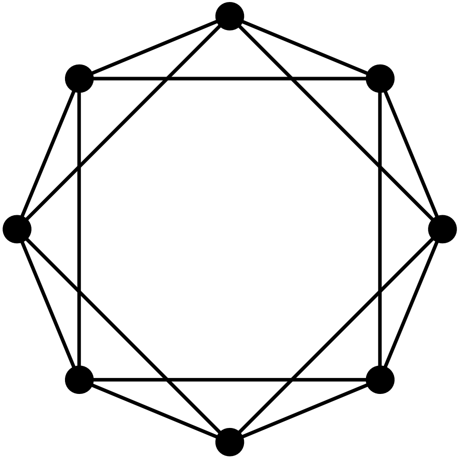
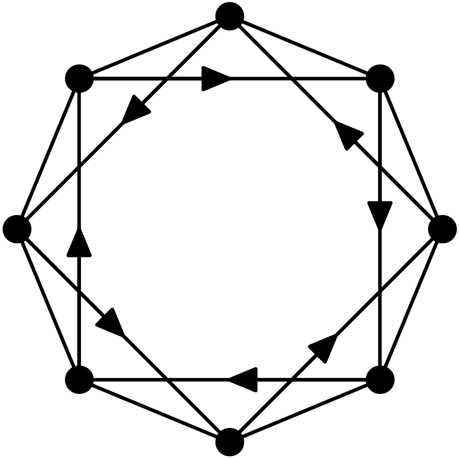
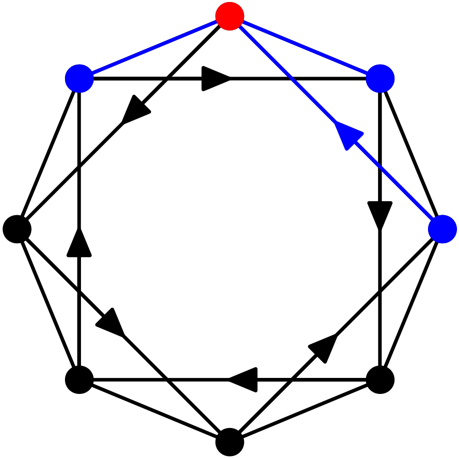
Logo of the conference



Modified logo



Modified logo - Indegree



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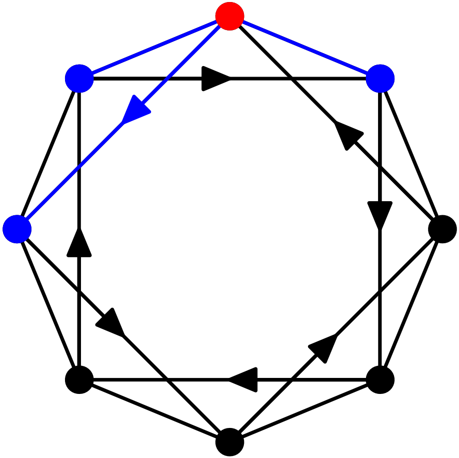
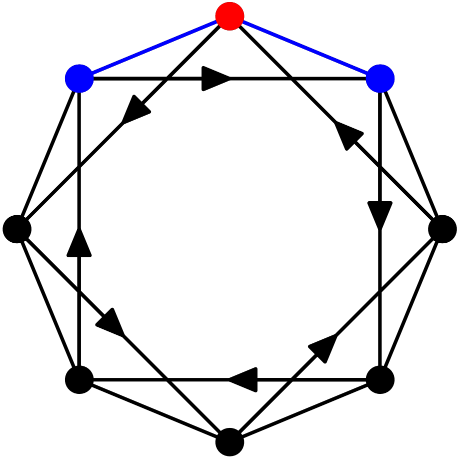
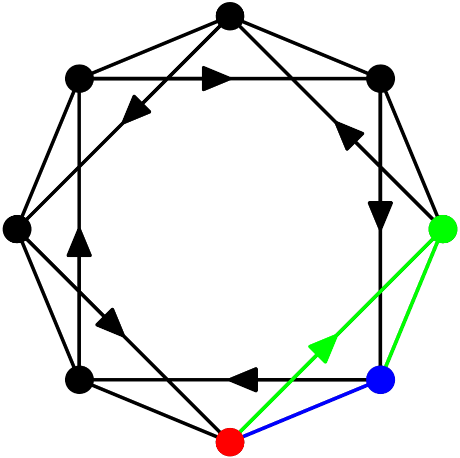


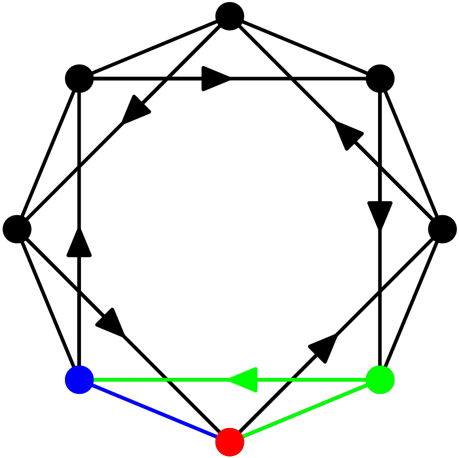
Modified logo - Regularity



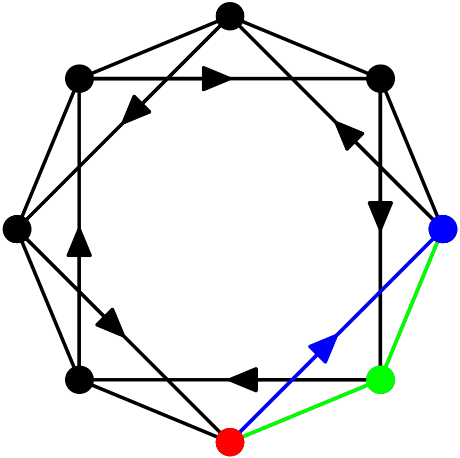
Modified logo - 2-walks



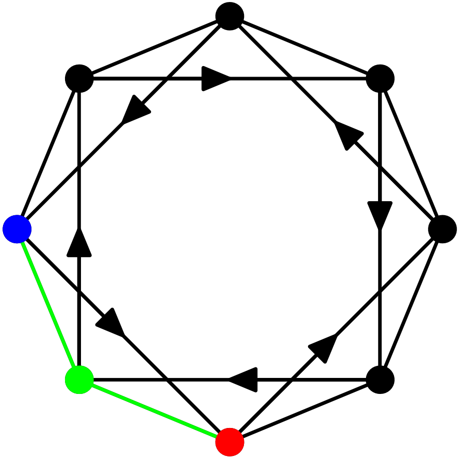
Modified logo - 2-walks



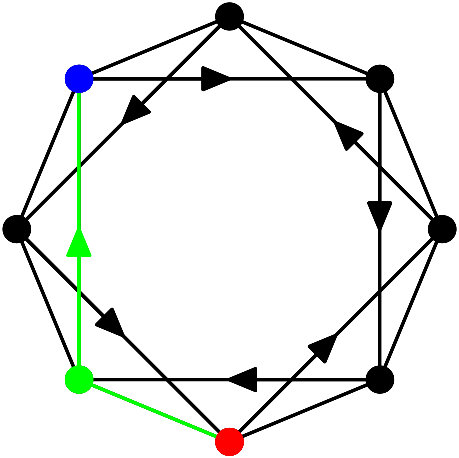
Modified logo - 2-walks



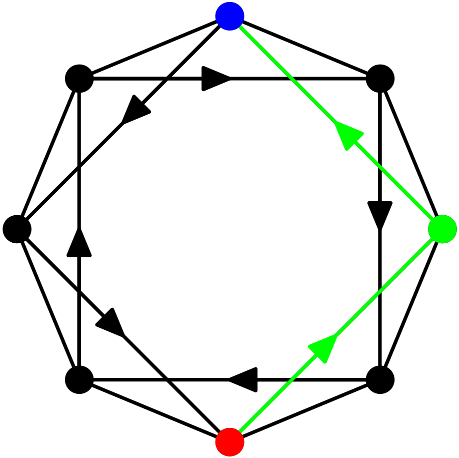
Modified logo - 2-walks



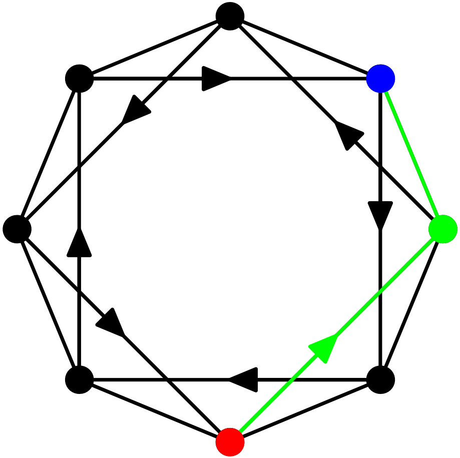
Modified logo - 2-walks



Modified logo - 2-walks



Modified logo - 2-walks



Modified logo - properties

Properties of the modified logo:

- It has $n = 8$ vertices.
- Each vertex has 3 out-neighbours and 3 in-neighbours, $k = 3$.
- Each vertex is incident to 2 undirected edges, $t = 2$.
- The number of 2-walks from any vertex u to any of its out-neighbour v, is $\lambda = 1$.
- The number of 2-walks from any vertex u to any of its non-out-neighbour v, is $\mu = 1$.

In other words, it is a directed strongly regular graph with parameters $(n, k, t, \lambda, \mu) = (8, 3, 2, 1, 1)$.

Directed strongly regular graphs

Definition (Duval, 1988)

Let $\Gamma = (V, D)$ be a directed graph, $|V| = n$, in which vertices have constant in- and out-degree k, but now only t edges being undirected. We say that Γ is a directed strongly regular graph with parameters (n, k, t, λ, μ) if there exist constants λ and μ such that the numbers of oriented uw-paths of length 2 are

1. t, if $u = w$;
2. λ, if $(u, w) \in D$;
3. μ, if $(u, w) \notin D$.

Directed strongly regular graphs

Due to 2-walk-regularity, the adjacency matrix A of a DSRG satisfies:

$$A^2 = tI + \lambda A + \mu(J - I - A),$$

where J is the all-one matrix, and I the identity matrix of the corresponding size.

Directed strongly regular graphs

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$(n, k, t, \lambda, \mu) = (6, 2, 1, 0, 1)$$

$$A^2 = 1 \cdot I + 0 \cdot A + 1 \cdot (J - I - A).$$

Directed strongly regular graphs

Remarks.

- If $t = 0$, then we are getting doubly regular tournaments.
- If $t = k$, then all the edges are undirected, so we are getting the classical strongly regular graphs.
- We will focus on the mixed case, i.e. when $0 < t < k$.

Directed strongly regular graphs

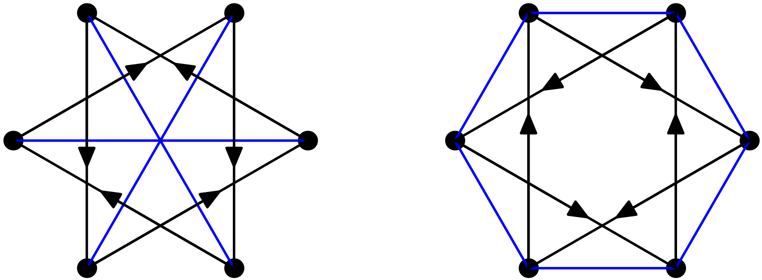


Figure: The smallest DSRGs appear with parameters $(6, 2, 1, 0, 1)$ and $(6, 3, 2, 1, 2)$.

Directed strongly regular graphs

Definition

Let $\Gamma = (V, D)$ be a digraph. Then its

- complementary digraph $\bar{\Gamma} = (\bar{V}, \bar{D})$ is defined on the same set of vertices $\bar{V} = V$ and the adjacency between vertices satisfies:

$$(u, v) \in \bar{D} \iff (u, v) \notin D.$$

- reverse digraph $\Gamma^T = (\tilde{V}, \tilde{D})$ is defined on the same set of vertices $\tilde{V} = V$ and the adjacency between vertices satisfies:

$$(u, v) \in \tilde{D} \iff (v, u) \in D.$$

Directed strongly regular graphs

Proposition (Duval, 1988)

If Γ is a DSRG with parameter set (n, k, t, λ, μ), then the complementary graph $\bar{\Gamma}$ is a DSRG with parameter set $(n, \bar{k}, \bar{t}, \bar{\lambda}, \bar{\mu})$, where

$$\begin{aligned}
\bar{k} &= n - k - 1 \\
\bar{t} &= n - 2k + t - 1 \\
\bar{\lambda} &= n - 2k + \mu - 2 \\
\bar{\mu} &= n - 2k + \lambda.
\end{aligned}$$

Directed strongly regular graphs

Proof.

If A is an adjacency matrix of a DSRG(n, k, t, λ, μ), then $A^2 = tI + \lambda A + \mu(J - I - A)$ and $AJ = JA = kJ$. Clearly, the adjacency matrix of its complement is $J - I - A$. Let us count

$$\begin{aligned}
(J - I - A)^2 &= J^2 - JI - JA - IJ + I^2 + IA - AJ + AI + A^2 \\
&= nJ - 2J - 2kJ + I + 2A + (tI + \lambda A + \mu(J - I - A)) \\
&= (n - 2k - 2 + \mu)J + (1 + t - \mu)I + (2 + \lambda - \mu)A \\
&= (n - 2k + t - 1)I + (n - 2k + \mu - 2)(J - I - A) + \\
&\quad + (n - 2k + \lambda)A.
\end{aligned}$$

$\square$

Proposition (Pech, 1997)

Let Γ be a DSRG. Then its reverse Γ^T is also a DSRG with the same parameter set.

Proof.

$$\begin{aligned}(A^T)^2 = A^T \cdot A^T = (A \cdot A)^T = (A^2)^T &= \\ = (tI + \lambda A + \mu(J - I - A))^T &= \\ = tI + \lambda A^T + \mu(J - I - A^T).\end{aligned}$$

$\square$

Duval's main theorem

Let Γ be a DSRG with parameters (n, k, t, λ, μ). Then there exists some positive integer d for which the following requirements are satisfied:

$$k(k + (\mu - \lambda)) = t + (n-1)\mu$$

$$(\mu - \lambda)^2 + 4(t - \mu) = d^2$$

$$d \mid (2k - (\mu - \lambda)(n-1))$$

$$\frac{2k - (\mu - \lambda)(n-1)}{d} \equiv n - 1 \pmod{2}$$

$$\left| \frac{2k - (\mu - \lambda)(n-1)}{d} \right| \leq n - 1.$$

Directed strongly regular graphs

Further necessary conditions

$$\begin{array}{rcccl} 0 & \leq & \lambda < t & < & k \\ 0 & < & \mu \leq t & < & k \\ -2(k-t-1) & \leq & \mu - \lambda & \leq & 2(k-t). \end{array}$$

Directed strongly regular graphs

Theorem (Duval, 1988)

All the eigenvalues of a DSRG are integers.

Theorem (Duval, 1988)

There is no DSRG of prime order.

Directed strongly regular graphs

The smallest feasible parameter sets and numbers of non-isomorphic graphs realizing them:

n	k	t	λ	μ	$\exists$?
6	2	1	0	1	1
8	3	2	1	1	1
10	4	2	1	2	16
12	3	1	0	1	1
12	4	2	0	2	1
12	5	3	2	2	20
14	5	4	1	2	0
14	6	3	2	3	16495

Directed strongly regular graphs

Usually, the main goals concerning DSRG's are:

1. To find a DSRG realizing a "new" parameter set.
2. To prove a non-existence result.
3. To find an infinite family of DSRG's.

The most important data are collected on the webpage of A. Brouwer and S. Hobart: http://homepages.cwi.nl/~aeb/math/dsrg

Loops, quasigroups, Latin squares

Definition

A Latin square of order n is an $n \times n$ array with n different entries, such that each entry occurs exactly once in any row and in any column of the array.

A quasigroup is a set Q with a binary operation "$\cdot$" such that for all $a, b \in Q$ the equations $a \cdot x = b$ and $y \cdot a = b$ have a unique solution in Q.

Loops, quasigroups, Latin squares

A loop L is a quasigroup with an identity element $e \in L$ with the property $e \cdot x = x \cdot e = x$ for every $x \in L$.

Example

The Cayley table of a loop of order 5.

$\star$	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	c	d	e	a
c	c	d	a	b	e
d	d	b	e	a	c

Construction 1.

Let $(Q, \cdot)$ be an arbitrary quasigroup of order $n \geq 2$.
Define a digraph Γ_1 of order $2n^2$, whose vertex set is
$V(\Gamma_1) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{Z}_2$.
The set $D(\Gamma_1)$ of darts is defined as follows:

- $(x, y, i) \mapsto (z, y, i)$ for all $i \in \mathbb{Z}_2$, $x, y, z \in \{1, 2, \ldots, n\}$, $x \neq z$;
- $(x, y, i) \mapsto (x, z, i)$ for all $i \in \mathbb{Z}_2$, $x, y, z \in \{1, 2, \ldots, n\}$, $y \neq z$;
- $(x, y, 0) \mapsto (xy, z, 1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, 1) \mapsto (z, yx, 0)$ for all $z \in \{1, 2, \ldots, n\}$.

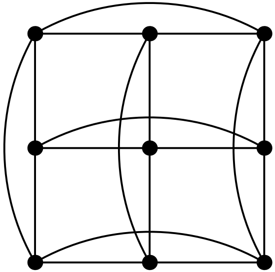
Theorem 1.

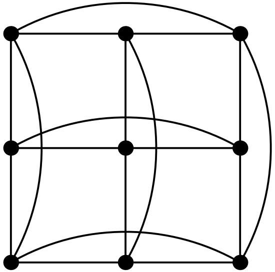
Γ_1 is a DSRG with parameter set $(2n^2, 3n - 2, 2n - 1, n - 1, 3)$.

Construction 1. - Example

Let us take $n = 3$ and the quasigroup with multiplication table

$\cdot$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

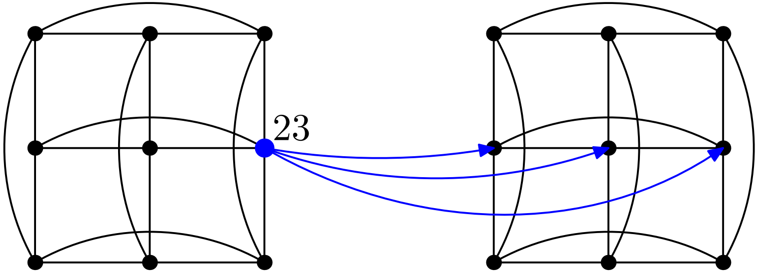




Construction 1. - Example

Let us take $n = 3$ and the quasigroup with multiplication table

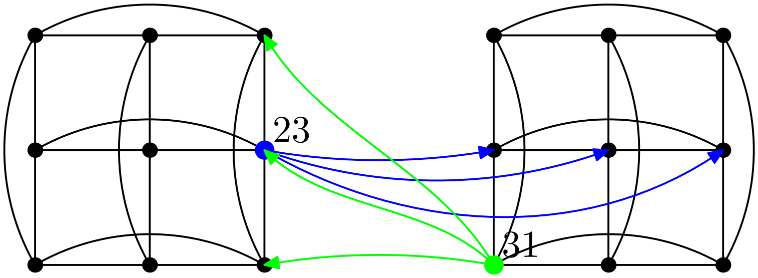
$\cdot$	1	2	3
1	1	2	3
2	3	1	**2**
3	2	3	1



Construction 1. - Example

Let us take $n = 3$ and the quasigroup with multiplication table

$\cdot$	1	2	3
1	1	2	**3**
2	3	1	**2**
3	2	3	1



Construction 2.

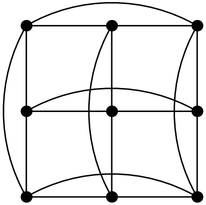
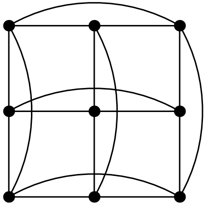
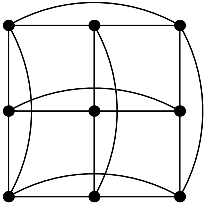
Let $(Q, \cdot)$ be an arbitrary quasigroup of order $n \geq 2$.
Define a digraph Γ_2 of order $3n^2$, whose vertex set is
$V(\Gamma_2) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{Z}_3$.
The set $D(\Gamma_2)$ of darts is defined as follows:

- $(x, y, i) \mapsto (z, y, i)$ for all $i \in \mathbb{Z}_3$, $x, y, z \in \{1, 2, \ldots, n\}$, $x \neq z$;
- $(x, y, i) \mapsto (x, z, i)$ for all $i \in \mathbb{Z}_3$, $x, y, z \in \{1, 2, \ldots, n\}$, $y \neq z$;
- $(x, y, i) \mapsto (xy, z, i+1)$ for all $i \in \mathbb{Z}_3$, and $z \in \{1, 2, \ldots, n\}$.
- $(x, y, i) \mapsto (z, yx, i-1)$ for all $i \in \mathbb{Z}_3$, and $z \in \{1, 2, \ldots, n\}$.

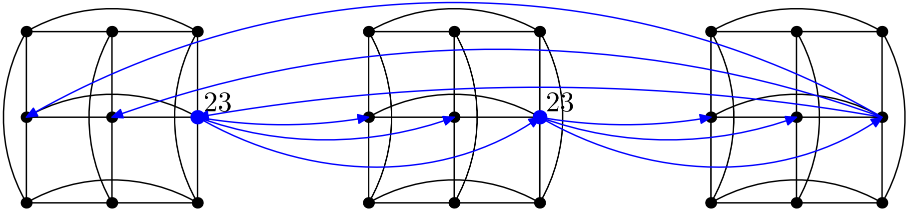
Theorem 2.

Γ_2 is a DSRG with parameter set $(3n^2, 4n-2, 2n, n, 4)$.

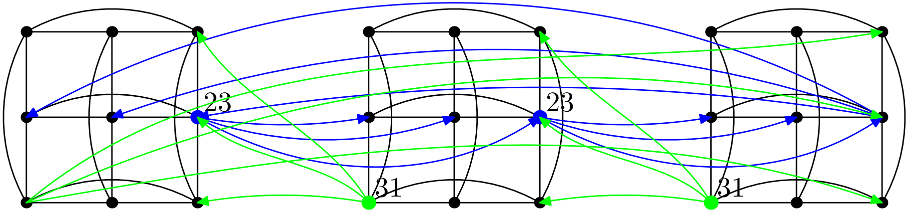
Construction 2. - Example

Construction 2. - Example



Construction 2. - Example



Construction 3.

Let $(L, \cdot)$ be an arbitrary loop of order $n \geq 2$, and c any non-identity element of L. Define a digraph Γ_3 of order $2n^2$, whose vertex set is $V(\Gamma_3) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{Z}_2$.
The set $D(\Gamma_3)$ of darts is defined as follows:

- $(x, y, i) \mapsto (z, y, i)$ for all $i \in \mathbb{Z}_2$, $x, y, z \in \{1, 2, \ldots, n\}$, $x \neq z$;
- $(x, y, i) \mapsto (x, z, i)$ for all $i \in \mathbb{Z}_2$, $x, y, z \in \{1, 2, \ldots, n\}$, $y \neq z$;
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- $(x, y, 1) \mapsto (z, yx, 0)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, 0) \mapsto (c(xy), z, 1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, 1) \mapsto (z, (yx)c, 0)$ for all $z \in \{1, 2, \ldots, n\}$.

Theorem 3.

Γ_3 is a DSRG with parameter set $(2n^2, 4n-2, 2n+2, n+2, 6)$.

Construction 4.

Let $(L, \cdot)$ be an arbitrary loop of order $n \geq 2$, and c any non-identity element of L. Define a digraph Γ_4 of order $3n^2$, whose vertex set is $V(\Gamma_4) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{Z}_3$.
The set $D(\Gamma_4)$ of darts is defined as follows:

- $(x, y, i) \mapsto (z, y, i)$ for all $i \in \mathbb{Z}_3$, $x, y, z \in \{1, 2, \ldots, n\}$, $x \neq z$;
- $(x, y, i) \mapsto (x, z, i)$ for all $i \in \mathbb{Z}_3$, $x, y, z \in \{1, 2, \ldots, n\}$, $y \neq z$;
- $(x, y, i) \mapsto (xy, z, i+1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, i) \mapsto (z, yx, i-1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, i) \mapsto (c(xy), z, i+1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, i) \mapsto (z, (yx)c, i-1)$ for all $z \in \{1, 2, \ldots, n\}$.

Theorem 4.

Γ_4 is a DSRG with parameter set $(3n^2, 6n-2, 2n+6, n+6, 10)$.

Proof.

- The parameters n, k and t can be computed easily.
- The proof of existence of λ and μ can be transformed into computations in the quasigroup Q, the main ingredient is counting of the number of solutions of various equations in Q.

Constructions 1 – 4. - Properties

n	2	3	4	5	6	7	8
# isom. classes of loops	1	1	2	6	109	23,746	106,228,849
# non-iso. DSRGs Con.2	1	1	2	6	109	23,746	???

Remark.

Two loops L_1 and L_2 are isomorphic if one can be obtained from the other by permutations of

- rows
- columns
- symbols
- roles of rows, columns and symbols.

Constructions 1 – 4. - Properties

Conjecture 1.

If we start from non-isomorphic loops L_1 and L_2 in Construction 2, then the corresponding DSRGs are also non-isomorphic.

Corollary.

The number of DSRGs grows hyper-exponentially in n for some parameter sets.

Remark.

Similar result for SRGs is already known due to Wallis and Fon-der-Flaass.

Constructions 1 – 4. - Properties

Conjecture 2.

The full automorphism group of the graph from Construction 2 is 6-times larger than the full automorphism group of the corresponding loop.

Lifts

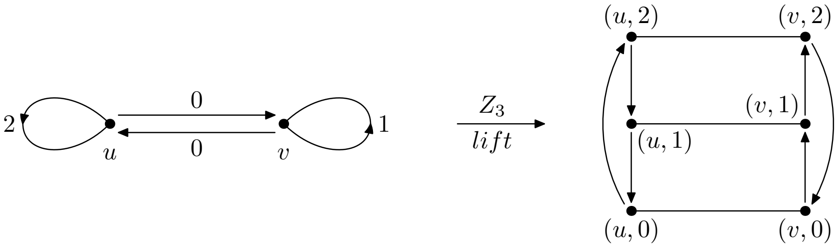

Figure: The smallest DSRG as a lift.

Lifts

Ingredients

- a group G
- base (multidi-)graph $\Gamma = (V, D)$
- function on darts $f : D \to G$

Lift

- Lifted graph Γ_f with
- vertex set $V(\Gamma_f) = V \times G$;
- edge/dart set $D(\Gamma_f) = D \times G$.

Darts in the lift

In the lift, (e, g) is a dart from the vertex (u, g) to the vertex (v, h) if and only if e is a dart from u to v in the voltage graph Γ and $h = gf(e)$.

Cayley graphs vs. lifts

Observation

If the base graph is a one-vertex graph, then the lift is a Cayley graph.

Let $e \notin X$, where e is the identity element of the group G. Then the digraph $\Gamma = \mathrm{Cay}(G, X)$ with vertex set G and dart set

$$\{(x, y) : x, y \in G, x^{-1}y \in X\}$$

is called the Cayley digraph over G with respect to X.

Cayley graphs

Question 1.

When is a Cayley graph $\mathrm{Cay}(G, X)$ a DSRG with parameters (n, k, t, λ, μ)?

Theorem [Klin, Munemasa, Muzychuk, Zieschang (2004)]

If $\Gamma = \mathrm{Cay}(G, X)$ is a DSRG, then G is a non-Abelian group.

Group rings

Definition

The group ring $\mathbb{Z}G$ is a free module over the group G with coefficients in $\mathbb{Z}$.

Example

Let G be the group $G = \langle a, b \mid a^2 = b^2 = 1, ab = ba \rangle$. Then the elements of $\mathbb{Z}G$ are of form

$$k_1 \cdot \underline{1} + k_2 \cdot \underline{a} + k_3 \cdot \underline{b} + k_4 \cdot \underline{ab},$$

where $k_1, k_2, k_3, k_4 \in \mathbb{Z}$.

Group rings

Example

Let

$$w_1 = 2 \cdot \underline{1} - 3 \cdot \underline{a} + 4 \cdot \underline{ab}$$

$$w_2 = 5 \cdot \underline{1} - 2 \cdot \underline{b} + 1 \cdot \underline{ab}.$$

Then

$$w_1 + w_2 = 7 \cdot \underline{1} - 3 \cdot \underline{a} - 2 \cdot \underline{b} + 5 \cdot \underline{ab}.$$

$$w_1 \cdot w_2 = 2 \cdot 5 \cdot \underline{1 \cdot 1} + 2 \cdot (-2) \cdot \underline{1 \cdot b} + 2 \cdot 1 \cdot \underline{1 \cdot ab} +$$

$$-3 \cdot 5 \cdot \underline{a \cdot 1} - 3 \cdot (-2) \cdot \underline{a \cdot b} - 3 \cdot 1 \cdot \underline{a \cdot ab} +$$

$$+4 \cdot 5 \cdot \underline{ab \cdot 1} + 4 \cdot (-2) \cdot \underline{ab \cdot b} + 4 \cdot 1 \cdot \underline{ab \cdot ab} =$$

$$= 10 \cdot \underline{1} - 4 \cdot \underline{b} + 2 \cdot \underline{ab} - 15 \cdot \underline{a} + 6 \cdot \underline{ab} - 3 \cdot \underline{b} + 20 \cdot \underline{ab} - 8 \cdot \underline{a} + 4 \cdot \underline{1} =$$

$$= 14 \cdot \underline{1} - 23 \cdot \underline{a} - 7 \cdot \underline{b} + 28 \cdot \underline{ab}.$$

Cayley graphs vs. lifts

Theorem [Hobart, Shaw (1999)]

When $|G| = n$, $|X| = k$ and

$$\underline{X} \cdot \underline{X} = t \cdot \underline{e} + \lambda \cdot \underline{X} + \mu \cdot \underline{G - e - X}$$

holds in the group ring $\mathbb{Z}G$, then $\mathrm{Cay}(G, X)$ is a DSRG(n, k, t, λ, μ).

Voltage assignments and DSRGs

Theorem [Gy., 2017]

Let G be the voltage group, Γ the base graph, M the voltage matrix over the group ring $\mathbb{Z}G$. Then the lift of Γ, with voltages as in the voltage matrix M, is DSRG with parameters (n, k, t, λ, μ), precisely when $n = |V(\Gamma)| \cdot |G|$, k is the number of elements in any row and column of M, and M satisfies

$$\underline{M} \cdot \underline{M} = t \cdot \underline{I} + \lambda \cdot \underline{M} + \mu \cdot \underline{J - I - M}$$

in the group ring $\mathbb{Z}G$, where I is the matrix with identities on the diagonal, and emptysets off-diagonal, and each entry of J is the group G.

Example 1.

The unique DSRG(8,3,2,1,1) can be obtained as a lift of a dipole over the voltage group $G = (\mathbb{Z}_4, \oplus)$, with voltage matrix

$$M = \begin{pmatrix} \{1\} & \{0,3\} \\ \{0,1\} & \{3\} \end{pmatrix}.$$

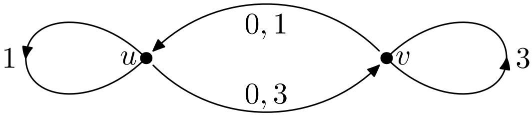

Figure: DSRG(8,3,2,1,1) as a lift.

Example 1.

$$\underline{M} \cdot \underline{M} = \begin{pmatrix} \underline{\{1\}} \oplus \underline{\{1\}} + \underline{\{0,3\}} \oplus \underline{\{0,1\}} & \underline{\{1\}} \oplus \underline{\{0,3\}} + \underline{\{0,3\}} \oplus \underline{\{3\}} \\ \underline{\{0,1\}} \oplus \underline{\{1\}} + \underline{\{3\}} \oplus \underline{\{0,1\}} & \underline{\{0,1\}} \oplus \underline{\{0,3\}} + \underline{\{3\}} \oplus \underline{\{3\}} \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \cdot \underline{0} + 1 \cdot \underline{\{1,2,3\}} & 1 \cdot \underline{\mathbb{Z}_4} \\ 1 \cdot \underline{\mathbb{Z}_4} & 2 \cdot \underline{0} + 1 \cdot \underline{\{1,2,3\}} \end{pmatrix} =$$

$$= 2 \cdot \begin{pmatrix} \{0\} & \emptyset \\ \emptyset & \{0\} \end{pmatrix} + 1 \cdot \begin{pmatrix} \{1\} & \{0,3\} \\ \{0,1\} & \{3\} \end{pmatrix} + 1 \cdot \begin{pmatrix} \{2,3\} & \{1,2\} \\ \{2,3\} & \{1,2\} \end{pmatrix} =$$

$$= 2 \cdot \underline{I} + 1 \cdot \underline{M} + 1 \cdot \underline{J - I - M}.$$

Example 2.

Leif K. Jørgensen:

There are 16 non-isomorphic DSRGs with parameters (10,4,2,1,2).

[Gy., 2017]

Two of them (+ their reverses) can be described as lifts of dipoles over the group $\mathbb{Z}_5$. Their voltage matrices are

$$M_1 = \begin{pmatrix} \{1,4\} & \{1,4\} \\ \{2,3\} & \{2,3\} \end{pmatrix} \qquad M_2 = \begin{pmatrix} \{1,2\} & \{0,4\} \\ \{0,1\} & \{3,4\} \end{pmatrix}.$$

Two more (10,4,2,1,2) DSRGs can be written as lifts, if we allow voltages from loops. For example, voltage matrix

$$\begin{pmatrix} \{a,b\} & \{a,b\} \\ \{c,d\} & \{c,d\} \end{pmatrix}$$

with voltages from the loop

$\star$	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	c	d	e	a
c	c	d	a	b	e
d	d	b	e	a	c

results in a DSRG(10,4,2,1,2), when voltages apply by right-multiplication.

Symmetry vs. regularity

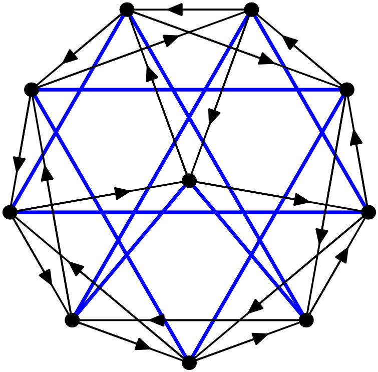

Figure: A self-reversed totally asymmetric DSRG(10,4,2,1,2).

$\mathrm{Aut}(\Gamma) = 1$, $rk(WL(\Gamma)) = 100$.

Bosák graph

Bosák graph is the unique DSRG(18,4,3,0,1).

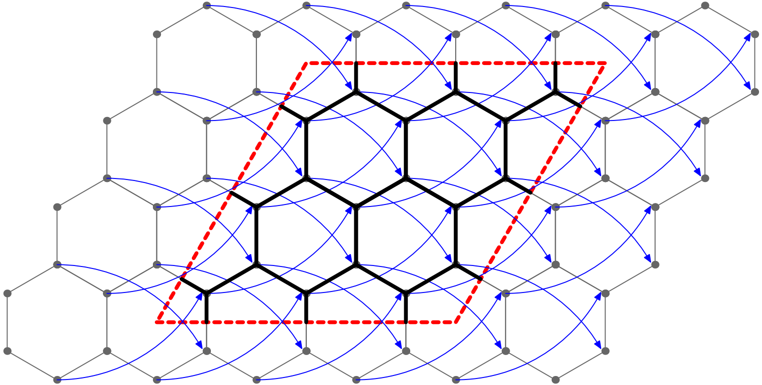

Figure: A drawing of the Bosák graph on torus.

The undirected part is the well-known Pappus graph.

Bosák graph

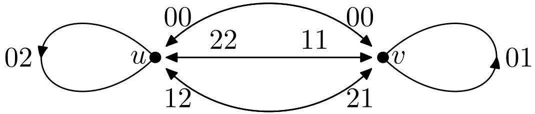

Figure: DSRG(18,4,3,0,1) as a lift of a dipole with voltage group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

- It contains 3 disjoint copies of DSRG(6,2,1,0,1).
- It is a mixed Moore-Cayley graph.

For more on mixed Moore-Cayley graphs see the talk of Sasha Gavrilyuk on Sunday, 19th August.

Thank you for your attention!