Regular Cayley maps for dihedral groups

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Maps

Definition. A (orientable) map $\mathcal{M}$ with an underlying connected graph Γ is a triple

$$\mathcal{M} = (\Gamma; R, L),$$

where R (rotation) is a permutation of the arc set $A(\Gamma)$, whose orbits are the sets of arcs initiated from the same vertex;

L (dart-reversing involution) is an involution of $A(\Gamma)$, whose orbits are the pairs of arcs based on the same edge.

Regular maps

Definition. An isomorphism $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a bijection $\varphi\colon A(\Gamma_1) \to A(\Gamma_2)$ such that

$$R_1\varphi = \varphi R_2 \text{ and } L_1\varphi = \varphi L_2.$$

Definition. An (orientation preserving) automorphism of a map $\mathcal{M} = (\Gamma; R, L)$ is an automorphism φ of Γ whose action on $A(\Gamma)$ induces an automorphism of $\mathcal{M}$.

Fact. For a map $\mathcal{M} = (\Gamma; R, L)$, the automorphism group $\mathrm{Aut}(\mathcal{M})$ acts semiregularly on $A(\Gamma)$, therefore, $|\mathrm{Aut}(\mathcal{M})| \leq |A(\Gamma)|$.

Definition. A map $\mathcal{M}$ is (orientably) regular if $|\mathrm{Aut}(\mathcal{M})| = |A(\Gamma)|$.

Cayley maps

Definition. Let X be a subset of a group G, $1 \notin X$, $X = X^{-1}$, and $\langle X \rangle = G$. The Cayley graph $\mathrm{Cay}(G, X)$ with connection set X is the graph (V, E) such that

$$V = G \text{ and } E = \{\, \{g, gx\} \mid g \in G,\ x \in X \,\}.$$

Definition. Let $\Gamma = \mathrm{Cay}(G, X)$ and p be a cyclic permutation of S. The Cayley map $\mathrm{CM}(G, X, p)$ is the map $\mathcal{M} = (\Gamma; R, L)$, where R is defined by

$$R : (g, gx) \mapsto (g, gp(x)),\ \ g \in G,\ x \in X.$$

Fact. If $\mathcal{M} = \mathrm{CM}(G, X, p)$, then $L(G) \le \mathrm{Aut}(\mathcal{M})$, where $L(G) = \{L_g : g \in G\}$, and L_g is the left translation: $L_g(x) = gx$, $x \in G$.

Regular Cayley maps and skew-morphisms

Suppose that $\mathcal{M} = \mathrm{CM}(G, X, p)$ is regular and $|X| = d$.

Then, $\mathrm{Aut}(\mathcal{M}) = L(G)\langle\varphi\rangle$, where φ is a generator of the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathcal{M})$.

For any $g \in G$,

$$\varphi L_g = L_{\varphi(g)}\varphi^{\pi(g)}, \text{ for some } \pi(g) \in \{0, 1, \ldots, d-1\}.$$

And so for any $h, g \in G$,

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h).$$

Since $R\varphi = \varphi R$, in the action of φ on the arcs of $\mathrm{Cay}(G, X)$, it follows that the restriction of φ to X is equal to p^i for some $i, \gcd(i, d) = 1$.

Regular Cayley maps and skew-morphisms

Definition (Jajcay, Širáň, 2002). A skew-morphism of a group G is a permutation φ of G such that there exists a function $\pi\colon G \to \{0, 1, \ldots, d-1\}$ (power function), where d is the order of φ, such that

- $\varphi(1) = 1$,
- $\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$ for all $g, h \in G$.

Note that, skew-morphisms are generalization of group automorphisms.

Regular Cayley maps and skew-morphisms

Theorem (Jajcay, Širáň, 2002)

A Cayley map $\mathrm{CM}(G, X, p)$ *is regular if and only if there exists a skew-morphism* φ *of G such that*

$$p(x) = \varphi(x) \text{ for every } x \in X.$$

The skew-morphism given in the above theorem is unique, and it is called skew-morphism associated with $\mathcal{M}$.

An arbitrary skew-morphism of G is associated with a regular Cayley map for G if and only if there exists a G-orbit X such that $\langle X \rangle = G$ and $X = X^{-1}$.

Regular Cayley maps and skew-morphisms

Definition. The kernel of a skew-morphism φ of G is defined by $\ker(\varphi) = \{x \in G : \pi(x) = 1\}$.

Properties.

- $\ker(\varphi)$ is a subgroup of G.
- The index $|G : \ker(\varphi)| < d$, where d is the order of φ, and $\pi(g) = \pi(h)$ if and only if g and h are in the same right coset of $\ker(\varphi)$.
- For any $g, h \in G$ and any positive integer j,

$$\varphi^j(gh) = \varphi^j(g)\varphi^{\sum_{i=0}^{j-1} \pi(\varphi^i(g))}(h).$$

- For any $g, h \in G$, $\pi(gh) = \sum_{i=0}^{\pi(g)-1} \pi(\varphi^i(h))$.

The index $|G : \ker(\varphi)|$ is called the skew type of φ, the skew type of a regular Cayley map is defined to be the skew type of the associated skew-morphism.

Regular Cayley maps and skew-morphisms

Question 1

Classify the regular Cayley maps for a given family of groups.

Solved for cyclic groups (Conder and Tucker, 2014) and simple groups (Bachratý, Conder and Verret). Several papers on dihedral groups (Feng, K, Kwak, Kwon, Marušič, Muzychuk, Wang, Zhang).

The goal of this talk is to present the solution for dihedral groups.

Question 2

Classify the skew-morphisms of a given family of groups.

Motivation. Skew-morphisms of are related with complementary factorizations: $G = AB$, where $A \cap B = 1$ and B is cyclic. Completely solved for dihedral groups (Kan and Kwon).

Regular Cayley maps and skew-morphisms

Definition. Two Cayley maps $\mathcal{M}_i = \mathrm{CM}(G, X_i, p_i)$, $i = 1, 2$, are equivalent if there exists an automorphism $\sigma \in \mathrm{Aut}(G)$ such that

$$\sigma(X_1) = X_2 \text{ and } \sigma^{X_1} p_1 = p_2 \sigma^{X_1},$$

where σ^{X_1} is the restriction of σ to X_1.

Notation: $\mathcal{M}_1 \equiv \mathcal{M}_2$.

The automorphism σ induces an isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$, and thus equivalent Cayley maps are also isomorphic as maps.

Construction 1: t-balanced regular Cayley maps

Definition. A regular Cayley map $\mathcal{M} = \mathrm{CM}(G, X, p)$ is t-balanced for a positive integer t if $p(x)^{-1} = p^t(x^{-1})$ for all $x \in X$. In particular, if $t = 1$, then we say that $\mathcal{M}$ is balanced.

Theorem (Conder, Jajcay and Tucker, 2006)

Let $\mathcal{M} = \mathrm{CM}(G, X, p)$ be a regular Cayley map with skew-morphism φ and power-function π.

- *$\mathcal{M}$ is balanced if and only if φ is an automorphism of G.*
- *$\mathcal{M}$ is t-balanced for $t > 1$ if and only if $t^2 \equiv 1 \pmod{|X|}$, π takes on only two values: 1 and t, in particular,*

$$|G : \ker(\varphi)| = 2.$$

Furthermore, φ maps $\ker(\varphi)$ to itself, and its restriction to $\ker(\varphi)$ is an automorphism.

Construction 1: t-balanced regular Cayley maps

We set $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$, $A_n = \langle a \rangle$ (A-type elements), $B_n = D_n \setminus A_n$ (B-type elements).

Theorem (Wang and Feng, 2005)

The balanced regular Cayley maps for D_n are equivalent to the Cayley maps $\mathcal{M}_1(n, \ell) := \mathrm{CM}(D_n, X, p)$,

$$p = (b, ab, a^{\ell+1} b, \ldots, a^{1+\ell+\cdots+\ell^{k-1}} b),$$

ℓ is a positive integer such that $\gcd(\ell, n) = 1$, and k is the smallest positive integer that $1 + \ell + \cdots + \ell^k \equiv 0 (\mathrm{mod}\ n)$.

The skew-morphism: $\varphi(a^i) = a^{\ell i}$ and $\varphi(a^i b) = a^{\ell i + 1} b$.

Construction 1: t-balanced regular Cayley maps

Theorem (Kwak, Kwon and Feng, 2006)

The t-balanced regular Cayley maps for D_n with $t > 1$ are equivalent to the Cayley maps $\mathcal{M}(n, \ell, k) := \mathrm{CM}(D_n, X, p)$,

$$p = (b, a, a^{2k}b, a^{\ell}, a^{2k(1+\ell)}b, a^{\ell^2}, \dots, a^{2k(1+\ell+\cdots+\ell^{M(\ell)-2})}b, a^{\ell^{2M(\ell)-1}}),$$

$n \geq 4$ is even and ℓ and k are positive integers such that $M(\ell)$ is even, $\ell^{M(\ell)/2} \equiv -1 \pmod{n}$ and $2k^2(\ell + \cdots + \ell^{M(\ell)/2}) \equiv 1 - \ell \pmod{n}$.

$M(\ell)$ is the smallest positive integer i that $\ell^i \equiv 1 \pmod{n}$.

Construction 1: t-balanced regular Cayley maps

$\mathcal{M}(n, \ell, k) := \mathrm{CM}(D_n, X, p)$,

$$p = (b, a, a^{2k}b, a^{\ell}, a^{2k(1+\ell)}b, a^{\ell^2}, \ldots, a^{2k(1+\ell+\cdots+\ell^{M(\ell)-2})}b, a^{\ell^{2M(\ell)-1}}),$$

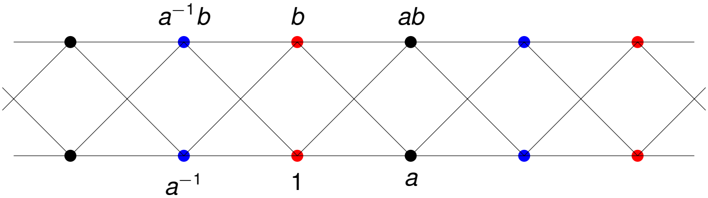
has skew-morphism

$$\varphi(a^i) = \begin{cases} a^{ri} & \text{if } 2 \mid i \\ a^{ri-r+u}b & \text{if } 2 \nmid i, \end{cases} \quad \varphi(a^i b) = \begin{cases} a^{ri+1} & \text{if } 2 \mid i \\ a^{ri-r+u+1}b & \text{if } 2 \nmid i. \end{cases}$$

where $r = -k(\ell + \ell^2 + \cdots + \ell^{M(\ell)/2})$ and $u = 2k$.

Construction 2: regular Cayley maps of skew-type 3

Let n be a positive integer divisible by 3.



$$N = \langle \mu_{\text{blue}}, \mu_{\text{red}} \rangle \cong \mathbb{Z}_2^2.$$

N is normalized by $L(D)$, let $G = N \rtimes L(D)$.

$G_1 = \langle L_b\, \mu_{\text{blue}} \rangle$, and so $L_b\, \mu_{\text{blue}}$ is a skew-morphism of D_n.

This induces the regular Cayley map

$$\mathcal{M}_3(n,1) := \mathrm{CM}(D_n, X, p),\ \ p = (a, a^{-1}, ab, a^{-1}b).$$

Construction 2: regular Cayley maps of skew-type 3

Let $\sigma \in \mathrm{Aut}(D_n)$ such that $\sigma(b) = b$ and $\sigma(a) = a^\ell$ with $M(\ell)$ is odd

$\ell \equiv 1 \pmod 3$, showing that N is centralized by σ, l et $G = (N \rtimes L(D_n)) \rtimes \langle\sigma\rangle$.

$G_1 = \big\langle L_b\, \mu_{\mathrm{blue}}, \sigma \big\rangle = \big\langle \sigma\, L_b\, \mu_{\mathrm{blue}} \big\rangle$, and so $\sigma L_b\, \mu_{\mathrm{blue}}$ is a skew-morphism of D_n.

This induces the regular Cayley map

$$\mathcal{M}_3(n,\ell) := \mathrm{CM}(D_n, X, p),\ \ p = (a, a^{-\ell}, a^{\ell^2} b, a^{-\ell^3} b, a^{\ell^4}, \ldots, a^{-\ell^{4M(\ell)-1}} b).$$

$\mathcal{M}_3(n,\ell)$ is not t-balanced and it has skew type 3.

Construction 2: regular Cayley maps of skew-type 3

Theorem (K, Marušič, Muzychuk, 2013)

The regular non-balanced Cayley maps for D_n with n odd are equivalent to Cayley maps $\mathcal{M}_3(n, \ell)$.

Theorem (Zhang, 2015)

The regular Cayley maps of skew type 3 are equivalent to the Cayley maps $\mathcal{M}_3(n, \ell)$, or

$$\mathcal{M}(D_6, X, p), \ p = (b, a, a^2 b, a^3, a^4 b, a^5).$$

Construction 2: regular Cayley maps of skew-type 3

Definition. For $B \leq A$, the core of B in A is the largest normal subgroup of A contained in B; and if this is trivial, then B is said to be core-free in A.

Theorem (K, Marušič and Muzychuk, 2011)

Let $\Gamma = \mathrm{Cay}(D_n, X)$ be a connected, G-arc-regular graph such that $L((D_n) \leq G$, and $L(A_n)$ is core-free in G. Then one of the following holds.

(i) $n = 1$, $\Gamma \cong K_2$ and $G \cong S_2$,

(ii) $n = 2$, $\Gamma \cong K_4$ and $G \cong A_4$,

(iii) $n = 3$, $\Gamma \cong K_{2,2,2}$ and $G \cong S_4$,

(iv) $n = 4$, $\Gamma \cong Q_3$ and $G \cong S_4$,

(v) $n = 2m$, m is an odd number, $\Gamma \cong K_{n,n}$ and $G \cong (D_n \times D_n) \rtimes \langle \sigma \rangle$, where σ is an automorphism of $D_n \times D_n$ interchanging the coordinates.

Construction 3: regular Cayley maps of skew-type $n/2$

Lemma

For $n \geq 2$, n is even, the Cayley map $\mathcal{M}_5(n, 1, 2) := \mathrm{CM}(D_n, X, p)$,

$$p = (b, a, a^2 b, a^3, a^4 b, \ldots, a^{n-2} b, a^{n-1})$$

is regular. The associated skew-morphism is

$$\varphi(a^i) = \begin{cases} a^{-i} & \text{if } 2 \mid i \\ a^{i+1} b & \text{if } 2 \nmid i, \end{cases} \quad \varphi(a^i b) = \begin{cases} a^{i+1} & \text{if } 2 \mid i \\ a^{-i} b & \text{if } 2 \nmid i; \end{cases}$$

it is of skew-type $n/2$, and $\ker(\varphi) = \langle a^{n/2}, a^{-1} b \rangle$.

Construction 3: regular Cayley maps of skew-type $n/2$

Lemma

For $n \geq 2$, n is divisible by 8, the Cayley map $\mathcal{M}_5(n, n/4+1, n/2+2) :=$ $\mathrm{CM}(D_n, X, p)$,

$$p = (b, a, a^{\frac{n}{2}+2}b, a^3, a^4b, \ldots, a^{\frac{n}{2}-2}b, a^{n-1})$$

is regular. The associated skew-morphism is

$$\varphi(a^i) = \begin{cases} a^{(\frac{n}{4}-1)i} & \text{if } 2 \mid i \\ a^{(\frac{n}{4}+1)i+\frac{n}{4}+1}b & \text{if } 2 \nmid i, \end{cases} \quad \varphi(a^ib) = \begin{cases} a^{(\frac{n}{4}+1)i+1} & \text{if } 2 \mid i \\ a^{(\frac{n}{4}-1)i+\frac{n}{4}}b & \text{if } 2 \nmid i; \end{cases}$$

it is of skew-type $n/2$, and $\ker(\varphi) = \langle a^{n/2}, a^{-1}b \rangle$.

Construction 3: regular Cayley maps of skew-type $n/2$

Theorem (K and Kwon, 2016)

Let $n \geq 2$, and $\mathcal{M} = \mathrm{CM}(D_n, X, p)$ be a regular Cayley map with associated skew-morphism φ and power function π. Then

- $\ker(\varphi)$ *is not contained in* A_n.
- $\mathcal{M}$ *is either the embedding of the octahedron into the sphere and* $|\ker(\varphi)| = 2$, *or* $|\ker(\varphi)| \geq 4$.

Conjecture (Conder, 2014)

The kernel of every skew-morphism of D_n is not contained in A_n.

The conjecture fails (Du and Zhang, 2016).

Construction 3: regular Cayley maps of skew-type $n/2$

Theorem (K and Kwon, 2016)

Let $\mathcal{M}$ be a regular Cayley map for D_n of skew-type $n/2$. Then exactly one of the following holds:

(i) $\mathcal{M} \equiv \mathrm{CM}(D_2, \{a, b, ab\}, (a, b, ab))$.

(ii) $\mathcal{M} \equiv \mathrm{CM}(D_4, \{a, a^{-1}, b\}, (a, b, a^{-1}))$.

(iii) $\mathcal{M} \equiv \mathrm{CM}(D_6, X,)\ p = (a, a^{-1}, ab, a^{-1}b)$.

(iv) $\mathcal{M} \equiv \mathcal{M}_5(n, 1, 2)$

(v) $\mathcal{M} \equiv \mathcal{M}_5(n, n/4 + 1, n/2 + 2)$.

Construction 3: regular Cayley maps of skew-type $n/2$

Proposition (K and Kwon, 2016)

Let $\mathcal{M}$ be a regular Cayley map for D_n such that the cyclic subgroup $L(A_n)$ is core-free in $\mathrm{Aut}(\mathcal{M})$. Then $\mathcal{M}$ is equivalent to one of the following Cayley maps:

(i) $\mathcal{M}_1 = \mathrm{CM}(D_1, \{b\}, (b))$;

(ii) $\mathcal{M}_2 = \mathrm{CM}(D_2, \{b, ab, a\}, (b, ab, a))$;

(iii) $\mathcal{M}_3 = \mathrm{CM}(D_3, \{a^{-1}, a, b, a^2b\}, (a^{-1}, a, b, a^2b))$;

(iv) $\mathcal{M}_4 = \mathrm{CM}(D_4, \{b, a, a^{-1}\}, (b, a, a^{-1}))$;

(v) $\mathcal{M}_5(n, 1, 2)$ *with* $n \equiv 2 \pmod 4$.

Construction 4: Mixing

Let $n \geq 4$ be even. For positive integers r, s, u, v such that $\gcd(r, n/2) = \gcd(s, n/2) = 1$, u is even and v is odd, define the permutation

$$\sigma_{r,s,u,v}(a^i) = \begin{cases} a^{si} & \text{if } 2 \mid i \\ a^{ri-r+u}b & \text{if } 2 \nmid i, \end{cases}$$

$$\sigma_{r,s,u,v}(a^i b) = \begin{cases} a^{i+1} & \text{if } 2 \mid i \\ a^{si-i+v}b & \text{if } 2 \nmid i. \end{cases}$$

The skew-morphism of $\mathcal{M}(n, \ell, k)$ (t-balanced map) is equal to $\sigma_{r,r,u,u+1}$.

The skew-morphisms of $\mathcal{M}_5(n, 1, 2)$ and $\mathcal{M}_5(n, n/4 + 1, n/2 + 2)$ (maps of skew-type $n/2$) are equal to $\sigma_{1,-1,2,-2}$ and $\sigma_{n/4+1,n/4-1,n/2+2,n/2-1}$, resp.

We constrcut new skew-morphisms (hence new regular Cayley maps) mixing these.

Construction 4: Mixing

$$\sigma_{r,s,u,v}(a^i) = \begin{cases} a^{si} & \text{if } 2 \mid i \\ a^{ri-r+u}b & \text{if } 2 \nmid i, \end{cases}$$

$$\sigma_{r,s,u,v}(a^i b) = \begin{cases} a^{i+1} & \text{if } 2 \mid i \\ a^{si-i+v}b & \text{if } 2 \nmid i. \end{cases}$$

It follows that $\sigma_{r,s,u,v}$ permutes the A_d-cosets for every $A_d \leq A_{n/2}$, $A_d = \langle a^{n/d} \rangle$.

The permutation of $D_{n/d}$ induced by $\sigma_{r,s,u,v}$ is $\sigma_{r',s',u',v'}$, where $r \equiv r', s \equiv s', u \equiv u'$ and $v \equiv v' \pmod{n/d}$.

Construction 4: Mixing

Let $n = 2^\alpha n_1$, n_1 is odd, and suppose there exist positive integers r, s, u and v satisfying the following conidtions:

(c1) There are even integers k and l such that

$$ur(1 + r^2 + \ldots + r^{k-2}) \equiv -2 (\mathrm{mod}\ n).$$

(c2) There exsits a factorization $n_1 = m_1 m_2$ such that $\gcd(m_1, m_2) = 1$ and

$$\begin{aligned}&\big(-s \equiv r \equiv 1 \ \text{ and } \ -v + 1 \equiv u \equiv 2 (\mathrm{mod}\ 2^\alpha m_1)\big) \ \text{ or}\\ &\big(r \equiv 2^{\alpha-2} m_1 + 1, \ \ s \equiv 2^{\alpha-2} m_1 = 1\\ &-v + 1 \equiv u \equiv 2^{\alpha-1} m_1 + 2 (\mathrm{mod}\ 2^\alpha m_1) + 2 \ \text{ with } \ \alpha \geq 3\big) \ \text{ and}\end{aligned}$$

$$r \equiv s, \ \ v \equiv u + 1, \ \ r^2 \equiv ur + 1 \ \text{ and } \ r^{2m} \equiv -1 (\mathrm{mod}\ 2m_2).$$

By (c2), $\sigma_{r,s,u,v} \equiv \sigma_{1,-1,2,-1}$ or $\sigma_{n'/4+1, n'/4-1, n'/2+2, n'/2-1} (\mathrm{mod}\ 2^\alpha m_1)$, where $n' = 2^\alpha m_1$, and $\sigma_{r,r,u,u+1} \equiv \sigma_{r,r,u,u+1} (\mathrm{mod}\ 2m_2)$.

Construction 4: Mixing

Lemma

Let $n = 2^\alpha n_1$, n_1 is odd, and suppose there exist positive integers r, s, u and v satisfying conidtions (c1) and (c2). Then $\sigma_{r,u,s,v}$ is a skew-morphism of D_n.

By the Chinese reminder theorem, s and v are determined by r and u. Denote by $\mathcal{M}_5(n, r, u)$ the regular Cayley map induced by $\sigma_{r,s,u,v}$.

The regular Cayley maps $\mathcal{M}_5(n, r, u)$ include both the t-balanced and the regular Cayley maps of skew type $n/2$.

Construction 4: Mixing

Example. $n = 30,\ m_1 = 3,\ m_2 = 5,\ r = 7,\ s = 2,\ u = 14,\ v = 5$.

$\gcd(r, 15) = \gcd(s, 15) = 1$.

(c1): $ru(1 + r^2 + \cdots + r^{k-2}) \equiv -2 \pmod{30}$ holds for $k = 10$

(c2): $-s \equiv r \equiv 1 \pmod 3,\ u \equiv 2 \pmod 6,\ v \equiv -1 \pmod 6;$
$r \equiv s \pmod 5,\ v \equiv u + 1 \pmod{10},$
$(ru + 1)^j \equiv -1 \pmod{10}$ holds for $j = 1$ and $r^2 \equiv ru + 1 \pmod{10}$.

$\mathcal{M}_5(30, 7, 14)$ is the regular map $R61.2$ in Marston Conder's database of regular maps. It is neither t-balanced nor of skew type 15.

Regular Cayley maps for D_n

Theorem (K and Kwon, 2018+)

Up to equivalence, the regular Caylay map for D_n are the following:

(i) $\mathcal{M}_1(n,\ell)$ *(balanced maps).*

(ii) $\mathcal{M}_2(n,r) = \mathrm{CM}(D_n, X, p)$ *(of skew type 2 non-t-balanced)*

$$p = (a, b, a^{r^2-r+n/2}b, a^{r^3}, a^{r^4-r}b, a^{r^5-r+n/2}b, \ldots, a^{r^{M(r)-1}-r+n/2}b),$$

$n \equiv 2 \pmod 4$, $6 \mid M(r)$ *and* $r^{M(r)/2} \equiv -1 \pmod n$.

(iii) $\mathcal{M}_3(n,\ell)$ *(of skew type 3)*

(iv) $\mathcal{M}_4(n,r) = \mathrm{CM}(D_n, X, p)$ *(of skew type 4)*

$$p = (b, a, a^{r+\frac{n}{2}}, a^{r^2-r^{-1}}b, a^{r^3}, a^{r^4+\frac{n}{2}}, \ldots, a^{r^{3M(r)-3}}, a^{r^{3M(r)-2}+\frac{n}{2}}),$$

$n \equiv 4 \pmod 8$, $3 \nmid M(r)$ *and* $r^{M(r)/2} \equiv \frac{n}{2} - 1 \pmod n$.

(v) $\mathcal{M}_5(n,r,u)$.

Sketch of proof

Proposition

Let $\mathcal{M} = \mathrm{CM}(G, X, p)$ *be a regular Cayley map with associated skew-morphism* φ, *and let* $N \leq G$ *be a normal subgroup such that* G/N *is a block system for* $\mathrm{Aut}(\mathcal{M})$. *Then*

(i) $\mathcal{M}/N := \mathrm{CM}(G/N, X/N, p^{G/N})$ *is a regular Cayley map.*

(ii) $\mathrm{Aut}(\mathcal{M}/N) = \mathrm{Aut}(\mathcal{M})^{G/N}$.

(iii) *The skew-morphisms associated with* $\mathcal{M}/N$ *is equal to* $\varphi^{G/N}$.

$\mathcal{M}/N$ is called the quotient of $\mathcal{M}$ by N.

Sketch of proof

We look for quotients by subgroups of $A_n = \langle a \rangle$.

Theorem (K and Kwon, 2016)

Let $\mathcal{M}$ be a regular Cayley map for D_n such that cyclic subgroup $L(A_n)$ is core-free in $\mathrm{Aut}(\mathcal{M})$. Then $\mathcal{M}$ is equivalent to one of the following Cayley maps:

(i) $\mathcal{M}_1 = \mathrm{CM}(D_1, \{b\}, (b))$;

(ii) $\mathcal{M}_2 = \mathrm{CM}(D_2, \{b, ab, a\}, (b, ab, a))$;

(iii) $\mathcal{M}_3 = \mathrm{CM}(D_3, \{a^{-1}, a, b, a^2b\}, (a^{-1}, a, b, a^2b))$;

(iv) $\mathcal{M}_4 = \mathrm{CM}(D_4, \{b, a, a^{-1}\}, (b, a, a^{-1}))$;

(v) $\mathcal{M}_5(n, 1, 2)$ *with* $n \equiv 2 \pmod 4$.

This implies that, if no quotient by non-trivial subgroups of A_n, then $\mathcal{M} \equiv \mathcal{M}_i$, $i = 1, 2, 3, 4$ or $\mathcal{M}_5 := \mathcal{M}_5(2, 1, 2) = \mathrm{CM}(D_2, \{a, b\}, (a, b))$.

Sketch of proof

Let $N < A_n$ be the largest subgroup such that D_n/A_d is a block system for Aut($\mathcal{M}$).

It follows that $\mathcal{M}/N \equiv \mathcal{M}_i$ for $i = 1, 2, 3, 4$ or 5.

- If $\mathcal{M}/N = \mathcal{M}_1$, then $\mathcal{M}$ is balanced, $\mathcal{M} \equiv \mathcal{M}_1(\ell)$ by earlier result.
- If $\mathcal{M}/N = \mathcal{M}_2$, then we prove that φ is of skew-type 2, find values of π, and derive that $\mathcal{M} \equiv \mathcal{M}_2(n, 1)$.
- If $\mathcal{M}/N = \mathcal{M}_3$, then we prove that $\mathcal{M}$ is of skew type 3, and $\mathcal{M} \equiv \mathcal{M}_3(n\ell)$ by earlier result.
- If $\mathcal{M}/N = \mathcal{M}_4$, then we prove that $\mathcal{M}$ is of skew type 4, find values of π, and derive that $\mathcal{M} \equiv \mathcal{M}_4(n, \ell)$.
- If $\mathcal{M}/N \equiv \mathcal{M}_5$, then we prove that $\mathcal{M} \equiv \mathcal{M}_5(n, r, u)$.

Sketch of proof: case $\mathcal{M}/N \equiv \mathcal{M}_5$

We reduce the problem for the case when n is a prime-power.

Lemma

If $\mathcal{M}/N \equiv \mathcal{M}_5$, then φ^2 is a skew-morphism of D_n, which normalizes the group $\langle L_{a^2}, L_b \rangle$.

Let ψ be the restriction of φ to $A_{n/2}$. ψ is a skew-morphism of $A_{n/2}$ for which $\psi^2 \in \mathrm{Aut}(A_{n/2})$ by the lemma.

Lemma

Let φ be a skew-morphism of a cyclic group G such that φ^2 is an automorphism of G. Then every subgroup $H \le G$ is a block for $L(G)\langle \varphi \rangle$.

Let $n/2 = p_1^{e_1} \cdots p_k^{e_k}$. Then $D/A_{n/2p^i} \cong D_{2p^i}$, and the above lemma allows us to define $\mathcal{M}/A_{n/(2p^i)}$.

Sketch of proof: case $\mathcal{M}/N \equiv \mathcal{M}_5$

$\mathcal{M}/A_{n/(2p^i)}$ is a regular Cayley map for D_{2p^i} such that

$$(\mathcal{M}/A_{n/(2p^i)})/(N/A_{n/(2p^i)}) \equiv \mathcal{M}_5.$$

Lemma

If $\mathcal{M}/N \equiv \mathcal{M}_5$ and $n = 2q^e$ for some prime q. Then

(i) $\mathcal{M}$ *is* $(d/2+1)$*-balanced, or*

(ii) $\mathcal{M} \equiv \mathcal{M}_5(n, 1, 2)$, *or*

(iii) $q = 2$, $e \geq 3$ *and* $\mathcal{M} \equiv \mathcal{M}_5(n, n/4+1, n/2+2)$.

Thank you for your attention.