PROLIFIC CONSTRUCTIONS OF STRICTLY DEZA GRAPHS

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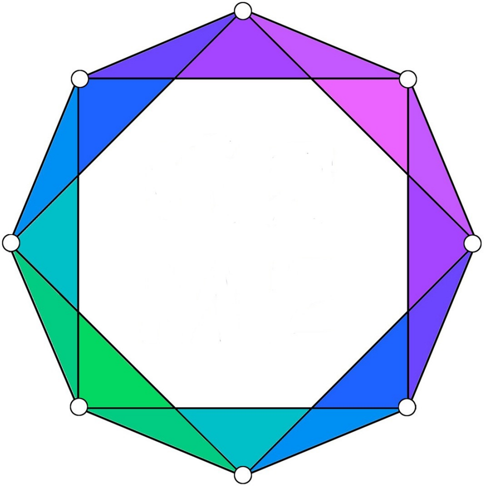
Definition 1. A non-empty k-regular graph Γ on n vertices is called a *strongly regular* with parameters (n, k, λ, μ), if the number of common neighbours of any two adjacent vertices is equal to λ and the number of common neighbours of any two distinct non-adjacent vertices is equal to μ.

Definition 2. A non-empty k-regular graph Γ on n vertices is called a *Deza graph with parameters* (n, k, b, a), where $n > k \geq b \geq a \geq 0$, if the number of common neighbours of any two distinct vertices takes the values a or b.

Definition 3. *A strictly Deza graph* is a Deza graph that is not a strongly regular graph and has diameter 2.

The conference logo is the smallest Deza graph.
Its parameters $(8, 4, 2, 1)$.



graph.jpg

The concept of Deza graphs was introduced in the initial paper:

[EFHHH] M. Erickson, S. Fernando, W. H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs. *J. Comb. Designs*. 7 (1999) P. 359–405.

In this paper a basic theory of strictly Deza graphs was developed and four constructions of such graphs were introduced. Moreover, all strictly Deza graphs with number of vertices at most 13 were found.

Later S. Goryainov and L. Shalaginov found all strictly Deza graphs whose number of vertices is equal to 14, 15, or 16. Also they found all strictly Deza graphs that are Cayley graphs with number of vertices less than 60.

[GSh1] S.V. Goryainov, L.V. Shalaginov, On Deza graphs with 14, 15 and 16 vertices, Siberian Electronic Math. Rep. 8 (2011) 105–115.
[GSh2] S.V. Goryainov, L.V. Shalaginov, Cayley–Deza graphs with less than 60 vertices, Siberian Electronic Math. Rep. 11 (2014) 268–310.

Some problems arising in the theory of strictly Deza graphs are similar to those in the theory of strongly regular graphs. However, results and methods in these theories sometimes differ, and an analysis of these differences can enrich both theories.

For example, recently G. Greaves and J. Koolen answered a question of A. Neumaier from 1981 about edge-regular graphs with regular clique. They found infinitely many examples of edge-regular graphs that have regular clique and that are not strongly regular. The smallest example in this series has 28 vertices.

A. Gavrilyuk suggested to check the list of small Cayley-Deza graphs found by S. Goryainov and L. Shalaginov in [GSh2] to find smaller example. It turns out that the list contains four non-isomorphic edge-regular strictly Deza graphs with parameters (24, 8, 4, 2), each of them has a 1-regular clique.

[GK] G.R.W. Greaves, J.H. Koolen, Edge-regular graphs with regular cliques, European Journal of Combinatorics 71 (2018) 194–201.

Strictly Deza graphs

Some links to bibliography on Deza graphs could be found in the homepage of Michel Marie Deza

https://web.archive.org/web/20120404180819/http://www.liga.ens.fr/

Adjacency matrix of Deza graphs

We can define Deza graphs in terms of matrices. Suppose Γ is a graph with n vertices, and M is its adjacency matrix. Then Γ is a Deza graph with parameters (n, k, b, a) if and only if

$$M^2 = aA + bB + kI$$

for some $(0,1)$-matrices A and B such that $A + B + I = J$, the all ones matrix. Note that Γ is a strongly regular graph if and only if A or B is M.

I. Construction from difference sets

Let G be a group and $D \subset G$. We will let D^{-1} denote the set $\{d^{-1} : d \in D\}$, and DD^{-1} denote the list $\{d_1 d_2^{-1} : d_1, d_2 \in D\}$ (i.e., DD^{-1} can have repeated elements). For disjoint subsets A and B of G, and integers a, b, we write

$$DD^{-1} = aA + bB + k\{e\}$$

if the list DD^{-1} contains a copies of each element of A, b copies of each element of B, and k copies of element e.

Let Γ be the graph with vertex set G, and vertices u and v be adjacent if and only if $v^{-1}u \in D$. Then Γ is a Deza graph with parametrs (n, k, b, a). In this case Γ is a Cayley graph.

II. Construction from strong product of graphs

The *strong product* $\Gamma_1[\Gamma_2]$ is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$, and adjacency defined by as follows:

(u_1, u_2) is adjacent to (v_1, v_2) iff u_1 is adjacent to v_1 or $u_1 = v_1$ and u_2 is adjacent to v_2.

Deza graphs with $b = k$

Theorem 1. ([EFHHH], Theorem 2.6) *Let Γ be a Deza graph with parameters (n, k, b, a). Then $k = b$ if and only if Γ is isomorphic to the strong product $\Gamma_1[\Gamma_2]$, where Γ_1 is a strongly regular graph with paramters (n_1, k_1, λ, μ), $\lambda = \mu$ and Γ_2 is $\overline{K_{n_2}}$ for some n_1, k_1, λ, n_2. Moreover, the parameters satisfy $n = n_1 n_2$, $k = k_1 n_2$, $a = \lambda n_2$, and $n_2 = \frac{k^2 - an}{k - a} \geq 2$.*

Deza graphs with $b = k$

The smallest example of such Deza graph we have from the triangular graph $T(6)$ with parameters $(15, 8, 4, 4)$ as Γ_1 and $\overline{K_2}$ as Γ_2.
W. D. Wallis [WW] and independent D. Fon-Der-Flaass [DF] gave a construction of strongly regular graphs with $\lambda = \mu$ using affine designs. It was D. Fon-Der-Flaass [DF] who has observed that construction gives rise to more than exponentially many strongly regular graphs with $\lambda = \mu$.

[DF] D. G. Fon-Der-Flaass, New prolific constructions of strongly regular graphs, Adv. Geom. 2 (2002), 301–306.

[WW] W. D. Wallis, Construction of strongly regular graphs using affine designs, Bull. Austral. Math. Soc. 4 (1971), 41–49.

III. Construction from dual Seidel switching

Let Γ be a strongly regular graph with parameters (n, k, λ, μ). Let M be the adjacency matrices of Γ.

Suppose that Γ has an automorphism φ of order two that interchanges only non-adjacent vertices.

Let P be the corresponding permutation matrix. Then $M' = PM$ is a symmetric matrix with zero diagonal.

So M' is the adjacency matrix of a graph Γ' (say), which is a Deza graph with the same set of parameters as Γ.

This extension of the construction was given by W.H. Haemers in [WH] and he has called the method *dual Seidel switching*.

[WH] W.H. Haemers, Dual Seidel switching, EUT Report 84-WSK-03, Eindhoven University of Technology, The Netherlands, 1984, pp. 183–190.

Affine designs

First, let us recall a prolific construction of strongly regular graphs was given by D.G. Fon-der-Flaass and in a somewhat more general form, it was given by W.D. Wallis.

As building blocks of the construction are used *affine designs*. Given a finite set V (of elements called points) and integers $k, p, \lambda \geq 1$, we define a affine design S to be a set $\mathcal{L}$ of k-element subsets of V, called blocks, such that any $v \in V$ is contained in p blocks, and any pair of distinct points u and v in V is contained in λ blocks.

So an affine design is a 2-design with the following two properties:

- every two blocks are either disjoint or intersect in a constant number r of points;
- each block together with all blocks disjoint from it forms a parallel class: a set of n mutually disjoint blocks partitioning all points of the design.

Affine designs

Examples of affine designs:

- all lines of an affine plane of order n, where $r = 1$;
- all hyperplanes of a d-dimensional affine space over the field $GF(n)$, where $r = n^{d-2}$;
- Hadamard 3-designs, where $n = 2$.

Parameters affine designs

All parameters of an affine design can be expressed in terms of r and n.

Let $(V, \mathcal{L})$ be an affine design with parallel classes of size n and block intersections of size r.
(i) The number $s = (r-1)/(n-1)$ is an integer.
(ii) The design has the following parameters:

- The number of points $v = |V| = n^2 r = n^3 s - n^2 s + n^2$;
- The number of blocks $b = |\mathcal{L}| = n^3 s + n^2 + n$;
- The number of parallel classes $p = |\mathcal{L}|/n = n^2 s + n + 1$;
- The block size $k = nr = n^2 s - ns + n$;
- $\lambda = ns + 1$.

The parameters satisfy the equality $k\lambda = (p-1)r$.

(iii) If $s = 0$ then the design is an affine plane of order n.

Here is a main construction due to by D.G. Fon-der-Flaass and even more general W.D. Wallis.

Construction. *Let* $i \in I = \{1, \ldots, p+1\}$. *Let* $S_i = (V_i, \mathcal{L}_i)$ *be arbitrary affine designs with parameters as in Lemma 1;* p *is the number of parallel classes in each* S_i.
For every i, *denote arbitrarily the parallel classes of* S_i *by symbols* $\mathcal{L}_{ij}$, $j \in I \setminus \{i\}$. *For* $v \in V_i$, *let* $l_{ij}(v)$ *denote the line in the parallel class* $\mathcal{L}_{ij}$ *which contains* v.

Construction of strongly regular graphs

For every pair i, j, $i \neq j$, choose an arbitrary bijection

$$\sigma_{ij} : \mathcal{L}_{ij} \to \mathcal{L}_{ji};$$

we only require that $\sigma_{ji} = \sigma_{ij}^{-1}$.

Construct a graph $\Gamma = \Gamma((S_i), (\sigma_{ij}))$ on the vertex set $X = \bigcup_{i \in I} V_i$.

The sets V_i will be independent.

Two vertices $v \in V_i$ and $w \in V_j, i \neq j$, are adjacent in Γ if and only if

$w \in \sigma_{ij}(l_{ij}(v))$ or, equivalently, $\sigma_{ij}(l_{ij}(v)) = l_{ji}(w)$.

The graph so obtained is strongly regular with parameters (V, K, Λ, M), $V = n^2r(n^2s + n + 2)$, $K = nr(n^2s + n + 1)$, $\Lambda = M = r(n^2s + n)$.

In particular, when we use affine planes of order n we get strongly regular graphs with parameters

$$(n^2(n + 2), n(n + 1), n, n).$$

We have the following observation by D.G. Fon-der-Flaass.

Given a fixed set of $n+2$ affine planes of order n, with fixed numberings of parallel classes according to Construction, and with fixed bijections σ_{1i} and σ_{2i} for all appropriate i, we can choose the bijections σ_{ij} for $i, j \geq 3$ in

$$n!^{\binom{n}{2}}$$

different ways. Form the list of all resulting graphs. Now, given an abstract graph Γ, how many times can it occur within the list?

For any set A of vertices, denote by $\mu(A)$ the set of their common neighbours. The graphs of Construction have the following property:

$(*)$ for any $x, y \in V_i$ the set $\mu(x, y)$ is a line in one of the planes S_j and $\mu(\mu(x, y))$ is the line in S_i through x and y.

Take three non-collinear points p_1, p_2, p_3 in S_1. Then, applying $(*)$ we uniquely find the sets of vertices corresponding to the lines p_1, p_2 of S_1. There are $(n-2)!(n-2)$ ways to assign vertices to all points of p_1, p_2, and to one more point of p_1, p_3. After this, repeated application of $(*)$ uniquely determines for each point of S_1 a vertex corresponding to it.

Also, the vertex sets corresponding to each line of each parallel class $\mathcal{L}_{j1}$ are now determined. In particular, the vertex sets corresponding to each S_i are determined. Further, for all $i \neq j$, the graph induced on $V_i \cup V_j$ determines the partition of V_i into the lines of the parallel class $\mathcal{L}_{ij}$. Finally, there are at most $n!$ ways to assign vertices to points of the line $\sigma_{12}(p_1, p_2)$. After this, every vertex of V_i, $i > 2$, is uniquely assigned to the intersection of two specified lines from the classes $\mathcal{L}_{i1}$ and $\mathcal{L}_{i2}$ So, all vertices of the graph get unique assignments, and in particular all permutations σ_{ij} are uniquely determined.

Strictly Deza graphs from affine designs

If we analyze D.G. Fon-der-Flaass's proof, then we can see that there are lots possibilities from Fon-der-Flaass–Wallis construction of strongly regular graphs to obtain automorphisms of order two and interchanges only adjacent vertices.

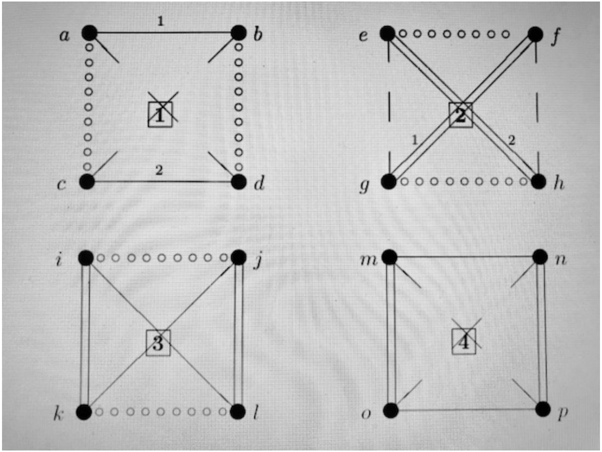
So we get strictly Deza graphs for complement of each strongly regular graph from their construction applying dual Seidel switching.

There are other generalizations of these construction of strongly regular graph.

P. J. Cameron and D. Stark A prolific construction of strongly regular graphs with the n-e.c. property, *The Electronic Journal of Combinatorics*, 9 (2002)

$(16, 6, 2, 2)$

Strongly regular graph with parameters $(16, 6, 2, 2)$.



Let Γ be a Deza graph with parameters (n, k, b, a). For a vertex x of Γ, put

$$A(x) := \{y \in V(\Gamma) : |N(x) \cap N(y)| = a\},$$

$$B(x) := \{y \in V(\Gamma) : |N(x) \cap N(y)| = b\},$$

Note that $|V(\Gamma)| = 1 + |A(x)| + |B(x)|$. Put

$$\beta(x) := |B(x)|.$$

It is known that $\beta(x)$ does not depend on the choice of a vertex. Moreover, $\beta(x)$ is uniquely determined by the parameters of Γ as follows:

$$\beta := \beta(x) = \frac{k(k-1)-a(n-1)}{b-a}.$$

The parameter β plays a key role in our investigation.

Theorem 2. ([KMSh]) *Let* Γ *be a strictly Deza graph with parameters* (n, k, b, a) *and* $\beta(\Gamma) > 1$. *The parameters* k *and* b *of* Γ *satisfy the condition* $k = b + 1$ *if and only if* Γ *is isomorphic to the strong product of* K_2 *with the complete multipartite graph with* $\frac{n}{n-k+1} > 1$ *parts of size* $\frac{n-k+1}{2}$.

Note that the adverb 'strictly' in Theorem 2 can not be removed, as is shown by the n-cycle with $n > 5$.

[KMSh] V.V. Kabanov, N.V. Maslova, L.V. Shalaginov, On strictly Deza graphs with parameters $(n, k, k-1, a)$, arXiv:1712.09529, December 2017.
Accepted to special issue of European Journal of Combinatorics dedicated to the memory of Michel Marie Deza.

Strictly Deza graphs with $b = k - 1$ and $\beta = 1$

Let Γ be a strictly Deza graph with parameters $(n, k, k-1, a)$ and $\beta = 1$. Then the vertices of Γ can be partitioned into pairs, where two vertices in every pair have b common neighbours, and two vertices from distinct pairs have a common neighbours.

This fact gives a connection with the notion of divisible design graphs.

[HKM] W.H. Haemers, H. Kharaghani, M.A. Meulenberg, Divisible design graphs, Journal of Combinatorial Theory, Series A. 118 (2011), 978–992.

We present two constructions of strictly Deza graphs with parameters $(n, k, k-1, a)$ and $\beta = 1$.

Both constructions use a strongly regular graph Δ with parameters (m, ℓ, λ, μ) where $\lambda = \mu - 1$.

First construction of Deza graphs with $\beta = 1$

Let Δ be a strongly regular graph with parameters (m, ℓ, λ, μ) where $\lambda = \mu - 1$.

Construction 1. *Let* Γ *be the strong product of* Δ *with* K_2. *The graph* Γ *is a Deza graph with parameters* $(n, k, k-1, a)$ *and* $\beta = 1$, *where* $n = 2\ell$, $k = 2\ell + 1$, $a = 2\mu$.

Second construction of Deza graphs with $\beta = 1$

Let Γ be the strong product of K_2 with Δ.

For any transposition $(x\ y)$ of the involution φ, modify Γ as follows:

- take the corresponding two pairs of vertices x', x'' and y', y'' in Γ
- delete the edges $\{x', x''\}$ and $\{y', y''\}$
- insert the edges $\{x', y''\}$ and $\{x'', y'\}$

Define Γ' to be the resulting graph. If A' is the adjacency matrix of Γ', then we can also construct Γ' from Γ with using dual Seidel switching as

$$A' = P_1 A, \text{ where } P_1 = P \otimes I_2.$$

We easily have that $(A')^2 = A^2$, which shows that Γ' is a Deza graph with the same parameters as Γ.

Second construction of Deza graphs with $\beta = 1$

Construction 2. *The graph* Γ' *is a Deza graph with parameters* $(n, k, k-1, a)$ *and* $\beta = 1$*, where* $n = 2m,\ k = 2\ell + 1,\ a = 2\mu$.

Note that in Γ any two vertices with b common neighbours are adjacent. For Γ' it is not true, therefore Γ and Γ' are non-isomorphic.

Theorem (S.V. Goryainov, W.H. Haemers, V.V. K., L.V. Shalaginov)
Let Γ *be a Deza graph with parameters* $(n, k, k-1, a)$, $k > 1$ *and* $\beta = 1$. *Then* Γ *can be obtained from Construction 1 or Construction 2.*

In case $k = 1$, Γ consists of $n/2$ disjoint edges and $\beta = 1$ implies $\Gamma = K_2$.

[GHKSh] S.V. Goryainov, W.H. Haemers, V.V. Kabanov, L.V. Shalaginov, Deza graphs with parameters $(n, k, k-1, a)$ and $\beta = 1$, arXiv:1806.03462, June 2018.
Submitted to Journal of Combinatorial Designs

Let q be a power of odd prime p and $q \equiv 1 \pmod 4$.

Let $\mathbb{F}_q$ be the field of order q and ω be a primitive root of $\mathbb{F}_q$.

Denote by S_1 the set of all even powers of ω in $\mathbb{F}_q$.

Definition. *The Paley graph $P(q)$* is the graph on $\mathbb{F}_q$ with two vertices being adjacent iff their difference belongs to S_1.

Denote by φ the automorphism of $P(q^2)$ that sends γ to γ^q. Note that φ fixes the elements from $\mathbb{F}_q$.

Lemma 2. *For any $\gamma = x + y\alpha$ from $\mathbb{F}_{q^2}$, the following holds:*
(1) $\gamma^q = x - y\alpha$;
(2) $\gamma - \gamma^q = 2y\alpha$.

Since α is a square iff $q \equiv 3(4)$, we have.

Lemma 3.
(1) If $q \equiv 1(4)$, then φ interchanges only non-adjacent vertices.
(2) If $q \equiv 3(4)$, then φ interchanges only adjacent vertices.

Thus, for any q, either the Paley graph $P(q^2)$ or its complement has an involution satisfying the condition of Construction 2.

Let Δ be a Paley graph $P(q^2)$ with the parameters $(4\mu+1, 2\mu, \mu-1, \mu)$, where $\mu = \frac{q^2-1}{4}$.

Take an order 2 automorphism of $P(q^2)$ that interchanges only nonadjacent vertices.

According to Construction 2, we obtain a new strictly Deza graph with parameters $(8\mu+2, 4\mu+1, 4\mu, 2\mu)$.

Hoffman-Singleton graph

The Hoffman-Singleton graph, which is strongly regular with parameters $(50, 7, 0, 1)$, has a unique involutive automorphism ϕ that interchanges only non-adjacent vertices.

The Deza graph obtained from Hoffman-Singleton graph with using dual Seidel switching, has diameter 3.

However, each of Constructions 1 and 2 produces a strictly Deza graph with parameters $(100, 15, 14, 2)$.

Construction 1 applied to the complement gives a strictly Deza graph with parameters $(100, 85, 84, 72)$.

Symmetric conference matrices

An $m \times m$ matrix C with zero's on the diagonal, and ± 1 elsewhere, is a *conference matrix* if $CC^\top = (m-1)I$. If a conference matric C is symmetric with constant row (and column) sum r, then $r = \pm\sqrt{m-1}$, and $B = \frac{1}{2}(J_m - I_m - C)$ is the adjacency matrix of a strongly regular graph with parameter set

$$\mathcal{P}(r) = (\ r^2+1,\ \tfrac{1}{2}(r^2-r),\ \tfrac{1}{4}(r-1)^2-1,\ \tfrac{1}{4}(r-1)^2\ ).$$

Note that $\mathcal{P}(-r)$ is the complementary parameter set of $\mathcal{P}(r)$.

Symmetric conference matrices with constant row sum were constructed by Seidel.

If q is an odd prime power and $r = \pm q$, then such a conference matrix can be obtained from the Paley graph of order q^2. Let B' be the adjacency matrix of $P(q^2)$, and put $S = J_{q^2} - I_{q^2} - 2B'$ (S is the so-called *Seidel matrix* of $P(q^2)$).

Define

$$C' = \begin{bmatrix} 0 & \mathbf{1}^\top \\ \mathbf{1} & S \end{bmatrix}$$

($\mathbf{1}$ is the all-ones vector). Then C' is a symmetric conference matrix of order $m = q^2 + 1$. However, C' doesn't have constant row sum.

Next we shall make the row and column sum constant by multiplying some rows and the corresponding columns of C' by -1.

This operation is called *Seidel switching*, and it is easily seen that Seidel switching doesn't change the conference matrix property.

To describe the required rows and columns, we use the notation and description of $P(q^2)$ given in the previous slides.

If $q \equiv 3 \mod 4$ we take the complement of the described Paley graph. Then the involution φ given in above slide interchanges only non-adjacent vertices in all cases.

For $x \in \mathbb{F}_q$ define $V_x = \{x + y\alpha \mid y \in \mathbb{F}_q\}$.

Then the sets V_x form a partition of the vertex set of $P(q^2)$, and each class is a coclique. Moreover, the partition is fixed by the involution φ.

Let V be the union of $\frac{1}{2}(q-1)$ classes V_x. Then V induces a regular subgraph of $P(q^2)$ of degree $\frac{1}{4}(q-1)^2-1$ with $\frac{1}{2}q(q-1)$ vertices.

Now make the matrix C by Seidel switching in C' with respect to the rows and columns that correspond with V. Then C is a regular symmetric conference matrix, and $B=\frac{1}{2}(J-I-C)$ is the adjacency matrix of a strongly regular graph Γ with parameter set $\mathcal{P}(q)$, and φ remains an involution that interchanges only nonadjacent vertices.

Thus, Γ satisfies the conditions of Constructions 1 and 2.

We obtain strictly Deza graphs with parameters

$$(q^2+1,\ \frac{1}{2}(q^2-q),\ \frac{1}{4}(q-1)^2,\ \frac{1}{4}(q-1)^2-1\ )$$

(by dual Seidel switching),

$$(2q^2+2,\ q^2-q+1,\ \frac{1}{2}(q^2-q),\ \frac{1}{2}(q-1)^2\ )$$

(by Construction 1 and 2),
and

$$(2q^2+2,\ q^2+q+1,\ \frac{1}{2}(q^2+q),\ \frac{1}{2}(q+1)^2\ )$$

(by Construction 1 applied to the complement).

If $q=3$, Γ is the Petersen graph. It has one conjugacy class of involutive automorphisms that interchanges only non-adjacent vertices. The Deza graph obtained from the Petersen graph with dual Seidel switching has diameter 3. However, for $q>3$, the obtained Deza graphs are strictly Deza.

Logo of the conference

The Paulus-Rozenfeld-Thompson graph T was independently discovered at least three times at Eindhoven (1973), Moscow (1973) and Tucson (1979). It is one of the ten strongly regular graphs with the parameters $(v, k, \lambda, \mu) = (26, 10, 3, 4)$.

Among these 10 graphs the SRG T has the largest group $G = Aut(T)$ of order 120, which is isomorphic to $A_5 \times Z_2$, the full symmetry group of the dodecahedron.

There exists one conjugacy class of involutions which interchanges only nonadjacent vertices. Hence, we have one strictly Deza graph from Construction 1 and one strictly Deza graph from Construction 2 with parameters $(52, 21, 10, 8)$. Also there exists one involution which interchanges only adjacent vertices. Hence, we have one strictly Deza graph from Construction 1 and one strictly Deza graph from Construction 2 with parameters $(52, 31, 18, 16)$.

Thank you for your attention!