On the semigroup of similarities with unique one point intersection, not satisfying WSP

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Self-similar sets

Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a system of contraction similarities in $\mathbb{R}^n$. Then the nonempty compact set $K = \bigcup\limits_{i=1}^{m} S_i(K)$ is called a *self-similar set*, generated by the system $\mathcal{S}$, or an *attractor* (*invariant set*) of the system $\mathcal{S}$;

K exists and unique for any such system $\mathcal{S}$ by Hutchinson's theorem[1].

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[1] J. Hutchinson, "Fractals and Self Similarity", Indiana University Mathematics Journal **30** (1981), 713–747

Open Set Condition (OSC)

The system $\mathcal{S}$ of contraction similarities is said to satisfy the *open set condition* (OSC), if there exists an open set O such that:
1) $S_i(O) \subset O$ for all $i \in I = \{1, \dots, m\}$;
2) $S_i(O) \cap S_j(O) = \varnothing$ for all distinct $i, j \in I$.

$\mathcal{F} = \{S_{\mathbf{j}}^{-1} S_{\mathbf{i}} : \ \mathbf{i}, \mathbf{j} \in I^*, \ \mathbf{i}$ and $\mathbf{j}$ are incomparable$\}$ is the *associated family of similarities*[2].

OSC $\Leftrightarrow \ \mathrm{Id} \notin \overline{\mathcal{F}}$.

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[2] Ch. Bandt, S. Graf, "Self-similar sets 7. A characterization of self-similar fractals with positive Hausdorff measure", Proc. Amer. Math. Soc. **114**:4 (1992), 995–1001

Open Set Condition (OSC)

OSC $\Rightarrow$ $d = \dim_H(K)$ is the solution of Moran's equation[3]:

$$\sum_{k=1}^{m} (\mathrm{Lip}\ S_i)^d = 1$$

OSC $\Leftrightarrow$ $H^d(K) > 0$ for d defined above.

The question: is it possible to have $\dim_H(K)$ equal to the solution of Moran's equation, but violate OSC?

[3] P. A. P. Moran, "Additive functions of intervals and Hausdorff measure", Proc. Cambridge Philos. Soc. **42** (1946), 15–23.

Weak Separation Property (WSP)

The system $\mathcal{S}$ has the *weak separation property* (WSP) iff[4]
$\mathrm{Id} \notin \overline{\mathcal{F} \setminus \mathrm{Id}}$.

OSC $\Rightarrow$ WSP, but the opposite is not true.

[4] M. P. W. Zerner, "Weak separation properties for self-similar sets", Proc. Amer. Math. Soc. **124**:11 (1996), 3529–3539

Critical set

A set $C(\mathcal{S}) = \bigcup_{i=1, j\neq i}^{m} S_i(K) \cap S_j(K)$ is called a *critical set* of the system $\mathcal{S} = \{S_1, \ldots, S_m\}$.

The violation of OSC is caused by overlaps of an attractor of system $\mathcal{S}$. If OSC does not hold, there is at least one point in C, but there is no guarantee that this point is unique – no such examples were constructed before.

Transversality method

Known methods like transversality method, allow to construct the systems $\mathcal{S}_p$, depending of parameter p, such that $\mathcal{S}_p$ does not satisfy WSP[5] [6] for Lebesgue-almost all p; but using this method we cannot control the type of overlaps.

[5] B. Barany, “On the Hausdorff dimension of a family of self-similar sets with complicated overlaps”, Fund. Math. **206** (2009), 49–59

[6] B. Barany, “Iterated function systems with non-distinct fixed points”, J. Appl. Math. Anal. Appl. **383**:1 (2011), 244–258.

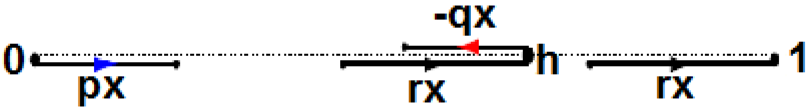
Our method, based on General Position Theorem, allows us to construct families of self-similar sets with prescribed behaviour of their critical sets.

For example, in previous work[7] we get exact overlap for double fixed points in 2-fold Cantor sets, which do not satisfy WSP.

In the current work we prove the existence of system with unique one point intersection, not satisfying WSP.

[7] K. Kamalutdinov, A. Tetenov, "Twofold Cantor sets in $\mathbb{R}$", 2018, https://arxiv.org/abs/1802.03872

Our example. Definition of $\mathcal{S}_{pqr}$

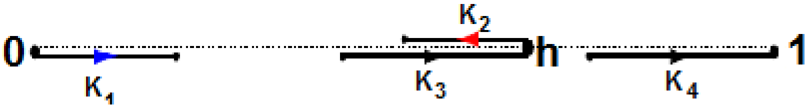


Take p, q, r in $(0, 1/16]$ and put $h = \dfrac{r + rq}{r + q}$.
We define the system $\mathcal{S} = \mathcal{S}_{pqr} = \{S_1, S_2, S_3, S_4\}$ of contraction similarities in $[0, 1]$:

$$S_1(x) = px,\ S_2(x) = h - qx,$$
$$S_3(x) = h - r + rx,\ S_4(z) = 1 - r + rx,$$

Let $K = K_{pqr}$ be an attractor of $\mathcal{S}$, and denote $K_i = S_i(K)$.

Our example. Unique one point intersection



Require that $q \in \left(\frac{r^2}{1-2r}, r \right)$, so $h < 1 - r$ and $h - r > (1-r)/2$, therefore the only possible non-empty intersection of the pieces of K is $K_2 \cap K_3$.

Our aim will be to find such values of p, q, r that

$$S_2(K) \cap S_3(K) = \{h\},$$

and we will say in this case that the system $\mathcal{S}$ has *unique one point intersection*.

Violation of WSP and dimension calculation

Proposition 1

If $\mathcal{S}$ has unique one point intersection, then it does not have WSP.

Theorem 2

If $\mathcal{S}_{pqr}$ has unique one point intersection, then Hausdorff dimension $d = \dim_H K_{pqr}$ satisfies Moran's equation $p^d + q^d + 2r^d = 1$, and Hausdorff measure $\mu^d(K_{pqr}) = 0$.

Existence problem

Our problem is how to find those p, q, r, for which each of the intersections

$$S_2 S_1^m S_i(K) \cap S_3 S_4^n S_j(K) \neq \varnothing \; (i \in \{2,3,4\},\ j \in \{1,2,3\})$$

is empty, so we analyse how large is the set of those triples (p, q, r) which do not possess such property.

General Position Theorem. Motivation

Let K be the attractor of a system $\mathcal{S} = \{S_1, ...S_m\}$ of contraction similarities in $\mathbb{R}^n$, and let $\dim_H K < n/2$. Suppose that $S_i(K) \cap S_j(K) \neq \varnothing$ for some $i, j \in \{1, \ldots, m\}^*$.

Is it possible to change the system $\mathcal{S}$ slightly to such system $\mathcal{S}' = \{S'_1, ...S'_m\}$, that its attractor K' satisfies the condition $S'_i(K) \cap S'_j(K) = \varnothing$?

General Position Theorem

Theorem 3

Let D, L_1, L_2 be compact metric spaces.
Let $\varphi_i(\xi, x) : D \times L_i \to \mathbb{R}^n$ *be continuous maps, such that*
(a) *they are α-Hölder with respect to x;*
(b) *there is $M > 0$ such that for any $x_1 \in L_1, x_2 \in L_2$, and any $\xi, \xi' \in D$ the function* $\Phi(\xi, x_1, x_2) = \varphi_1(\xi, x_1) - \varphi_2(\xi, x_2)$ *satisfies the inequality*

$$\|\Phi(\xi', x_1, x_2) - \Phi(\xi, x_1, x_2)\| \geq M|\xi' - \xi|$$

Then $\Delta = \{\xi \in D | \ \varphi_1(\xi, L_1) \cap \varphi_2(\xi, L_2) \neq \varnothing\}$ is a compact set such that

$$\dim_H \Delta \leq \min\left\{\frac{\dim_H L_1 \times L_2}{\alpha}, d\right\}$$

Applying General Position Theorem

Using General Position Theorem, we show that for any pair of non-negative $m, n \in \mathbb{Z}$ and for any $r \in (0, 1/16)$, $p \in (0, r)$ the set $\Delta_{mn}(p, r)$ of those $q \in (0, r)$, for which $S_2 S_1^m S_i(K) \cap S_3 S_4^n S_j(K) \neq \varnothing$ $(i \in \{2, 3, 4\}, j \in \{1, 2, 3\})$, has dimension less than 1.

Therefore for any $r \in (0, 1/16)$, $p \in (0, r)$ the set $\bigcup\limits_{m,n=0}^{\infty} \Delta_{mn}(p, r)$ of exceptional parameters q has zero 1-dimensional Lebesgue measure.

Parametrization

Denote $I = \{1, 2, 3, 4\}$. Let $0 < a < 1$ and I_a^∞ be the space I^∞ supplied with the metrics $\rho_a(\sigma, \tau) = a^{s(\sigma,\tau)}$, where $s(\sigma, \tau) = \min\{k : \sigma_k \neq \tau_k\} - 1$.
This metric turns I^∞ to a self-similar set having Hausdorff dimension $\dim_H I_a^\infty = -\dfrac{\log 4}{\log a}$. Particularly, if $0 < a < \dfrac{1}{16}$, then $\dim_H I_a^\infty < 1/2$.

We use the space I_a^∞ to parametrise the sets K and further apply this in General Position Theorem:

Parametrization

Let $\pi = \pi_{pqr} : I^\infty \to K_{pqr}$ be an *address map*, that is $\pi(i_1 i_2 \dots) = \lim_{k\to\infty} S_{i_1\dots i_k}(x)$ for any x.

Lemma 4

Let $p, q, r \in (0, a)$ and $a < \dfrac{1}{16}$. Then $\pi_{pqr} : I_a^\infty \to K$ is a 1-Lipschitz map.

Anti-Lipschitz inequality

We consider the following sets:

$$D_{mn}(p,r) = \left\{ q \in (0,r) : \ \frac{r^{n+1}}{p^m} < \frac{q+r}{1-r} \right\}.$$

Lemma 5

Let $\varphi_q(\sigma) = S_2 S_1^m S_i \pi_{pqr}(\sigma)$ and $\psi_q(\tau) = S_3 S_4^n S_j \pi_{pqr}(\tau)$, where $i \in \{2,3,4\}$, $j \in \{1,2,3\}$ and $p < r < 1/16$ are fixed. Then for any $\sigma, \tau \in I^\infty$ and for any $q, q' \in D_{mn}(p,r)$:

$$|\varphi_q(\sigma) - \psi_q(\tau) - \varphi_{q'}(\sigma) + \psi_{q'}(\tau)| > \frac{p^m}{5}|q - q'|$$

Applying General Position Theorem

Theorem 6
Let $p \in (0, r)$*,* $r \in (0, 1/16)$*. Then for any* $m, n \in \mathbb{N}$ *the set*

$$\Delta_{mn}(p, r) = \{q \in (0, r) : \; S_2 S_1^m S_i(K_{pqr}) \cap S_3 S_4^n S_j(K_{pqr}) \neq \varnothing\}$$

is closed and nowhere dense in $(0, 1/16)$*.*

How many "good" parameters exist

Theorem 7

Fix $r \in (0, 1/16)$. The set $\mathcal{K}_r$ of those $(p, q) \in (0, r) \times \left(\frac{r^2}{1-2r}, r\right)$, for which $\mathcal{S}_{pqr}$ has unique one point intersection, has full measure in $\mathcal{V}_r = (0, r) \times \left(\frac{r^2}{1-2r}, r\right)$, and its complement is uncountable and dense in $\mathcal{V}_r$.

Sources

1. *J. Hutchinson*, “Fractals and Self Similarity”, Indiana Univ. Math. J. **30** (1981), 713–747

2. *P. A. P. Moran*, “Additive functions of intervals and Hausdorff measure”, Proc. Cambridge Philos. Soc. **42** (1946), 15–23

3. *Ch. Bandt, S. Graf*, “Self-similar sets 7. A characterization of self-similar fractals with positive Hausdorff measure”, Proc. Amer. Math. Soc. **114**:4 (1992), 995–1001

4. *M. P. W. Zerner*, “Weak separation properties for self-similar sets”, Proc. Amer. Math. Soc. **124**:11 (1996), 3529–3539

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