A Penrose Coloring Formula for Non-Planar Graphs and State Models in Graph Theory and Topology

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Applications of Negative Dimensional Tensors

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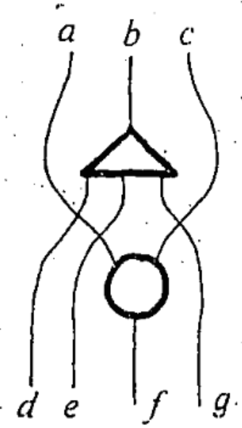
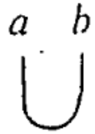
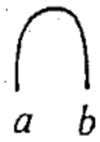


$= \theta^{ac}_{f}\, \chi^{b}_{deg}$

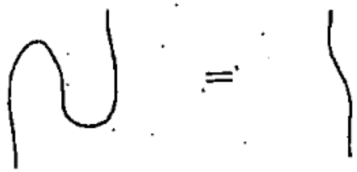
The "dimension" ν is depicted simply as a closed loop:

$$= \delta^a_a = \nu .$$

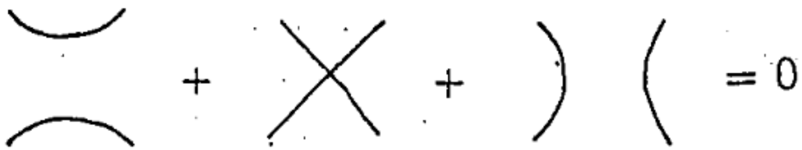
It now ceases to be important to maintain a distinction between upper and lower indices. For, writing

$$g^{ab} =$$  , $$g_{ab} =$$  ,

which is consistent with

 ,

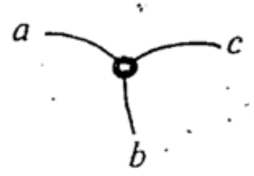
The most interesting system of all is, perhaps, the regular Cartesian system with $\nu = -2$. The elements of this system I call *binors*. The above basic identical relation can be written out as



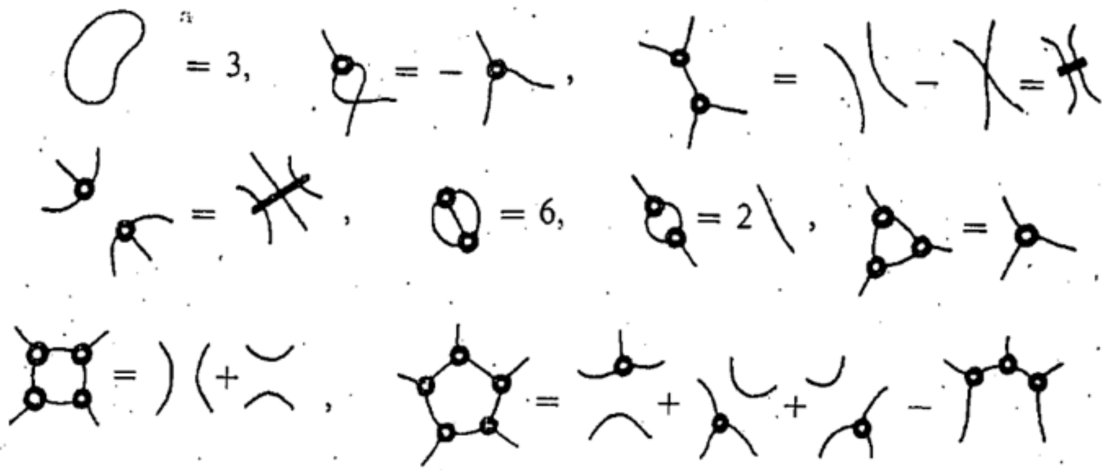
We readily see that every contraction of this expression is identically zero if $v = -2$. This expression must indeed vanish for a (weakly) regular $\{\overset{c}{\mathscr{I}}\}$-system with $\nu = -2$.

the remaining components being zero. Contraction is represented by the Einstein summation convention in the usual way. Depict:

$\delta_{ab} =$ 

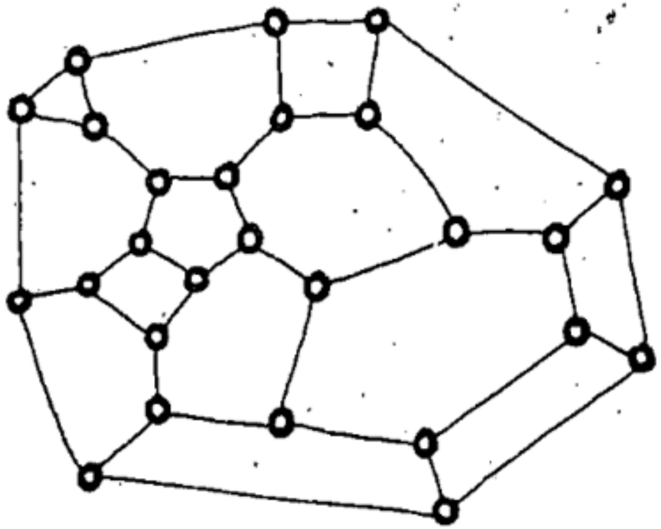
, $i\,\varepsilon_{abc} =$ 

where the factor $i\,(=\sqrt{-1})$ is included for simplicity in the signs of reduction formulae. We have, as well-known formulae of tensor calculus, or by repeated applications, thereof:



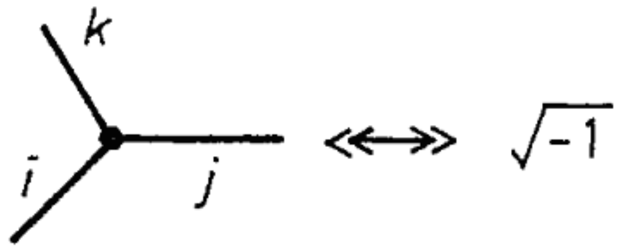
etc.†

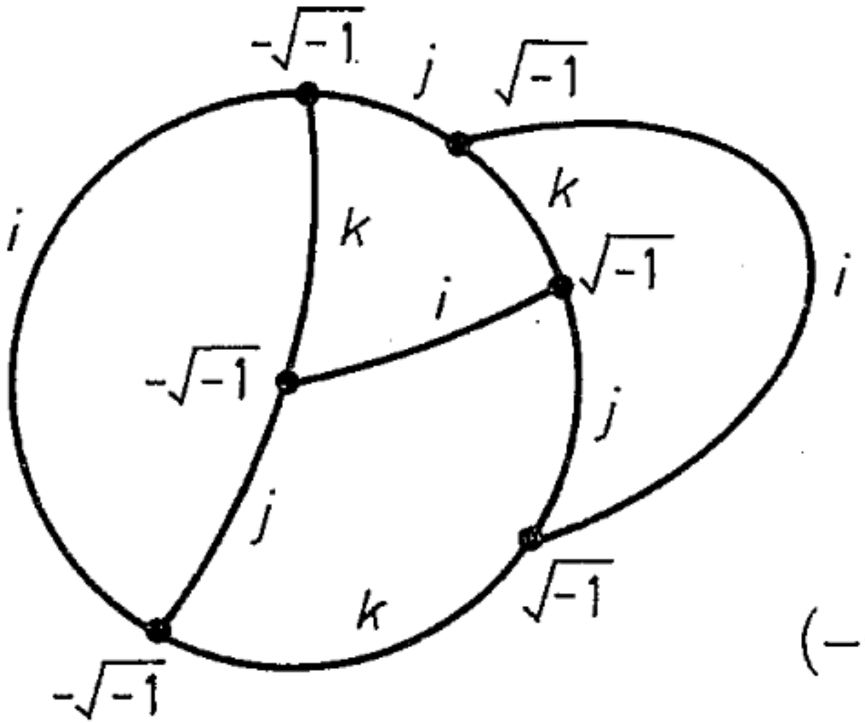
Now consider a planar graph of degree three. We can associate with it a certain contracted product of ε_{abc}'s, where one $i\varepsilon_{abc}$ is drawn at each vertex, a contraction occurring for each edge of the graph. The result is just some complex number (actually an integer). For example:

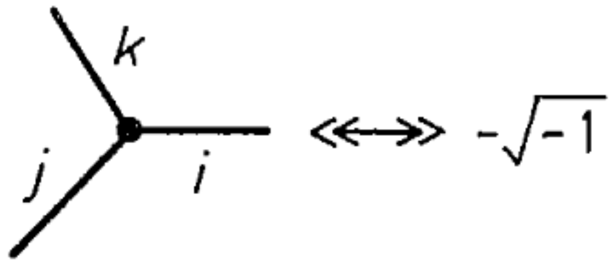
 $= K.$

The number K is, in fact, precisely the number of ways of colouring the edges of the graph with three colours so that three distinct colours occur at each

For each proper coloring, the product of the square roots of negative unity is equal to one.







$$\left(-\sqrt{-1}\right)^3\left(\sqrt{-1}\right)^3 = +1\ .$$

We will prove this fact, after some discussion.

Map Theorem for Cubic Graphs. A connected plane cubic graph without isthmus is properly edge-colorable with three colors.

We now introduce a diagrammatic representation for the coloring of a cubic graph. Let G be a cubic graph and let $C(G)$ be a coloring of G. Using the colors r, b and p we will write purple as a formal product of red and blue:

$$p = rb.$$

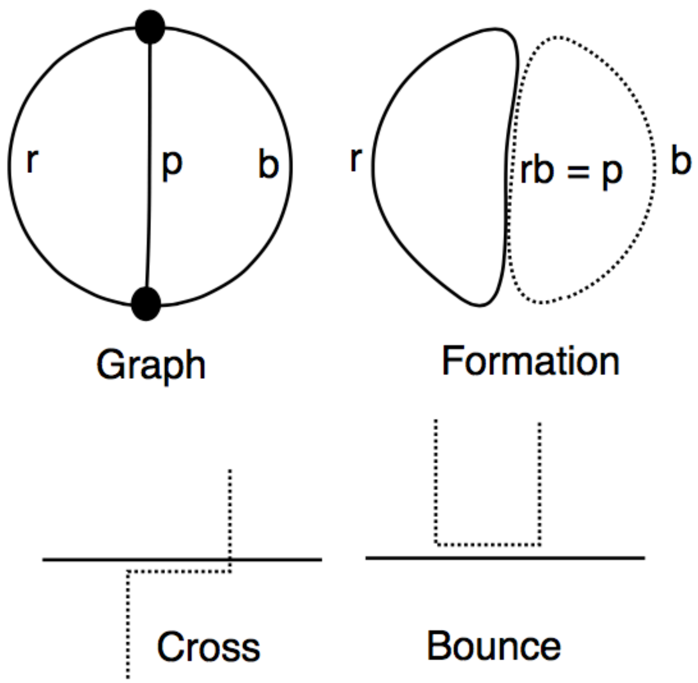
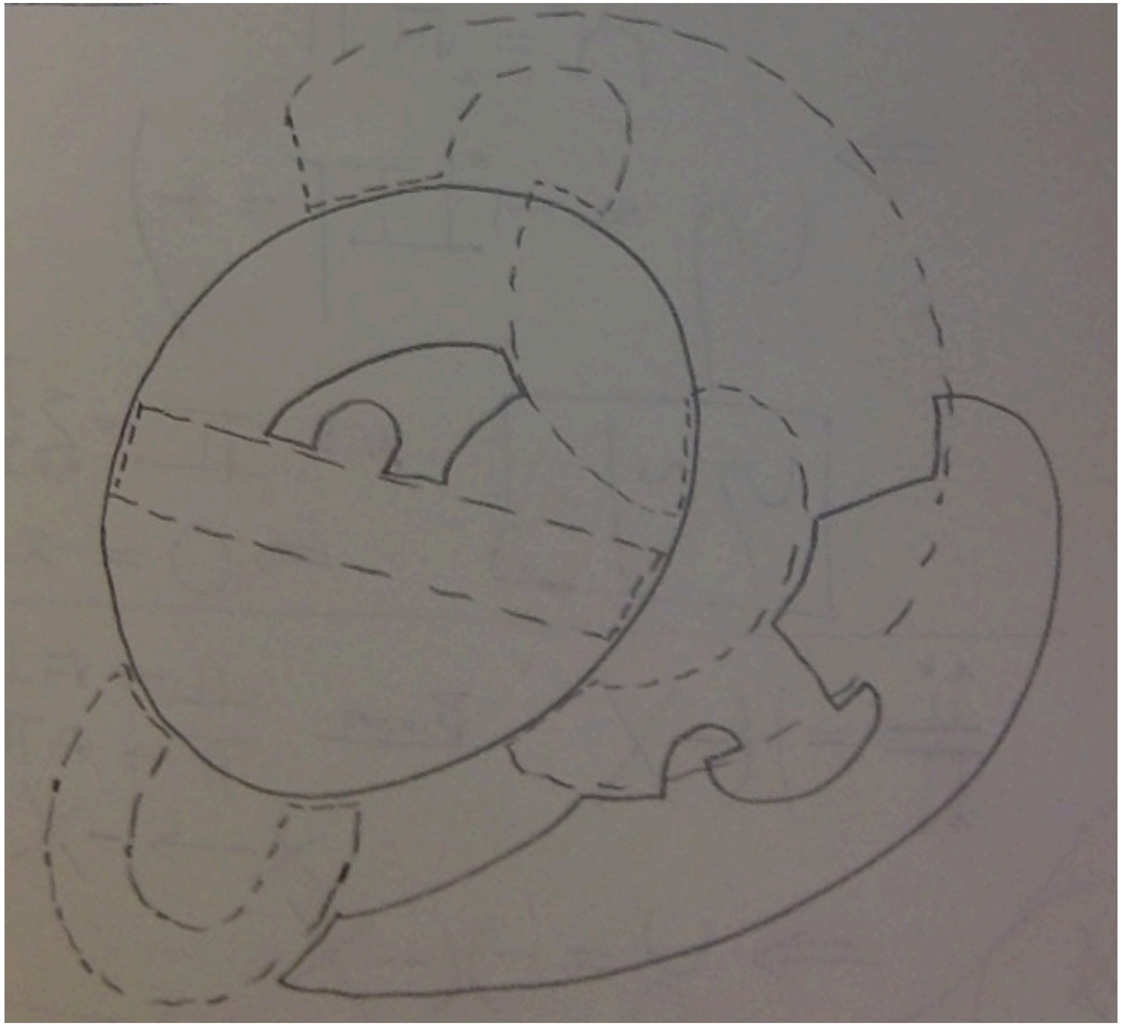


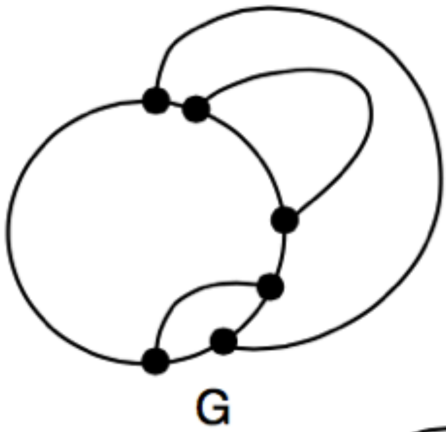
Figure 1: **Coloring and Formation.**

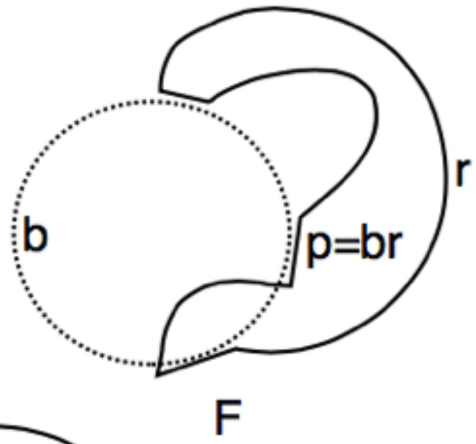
FourColorTheorem is Equivalent to Every Isthmus -Free Planar Cubic Graph has a Formation.

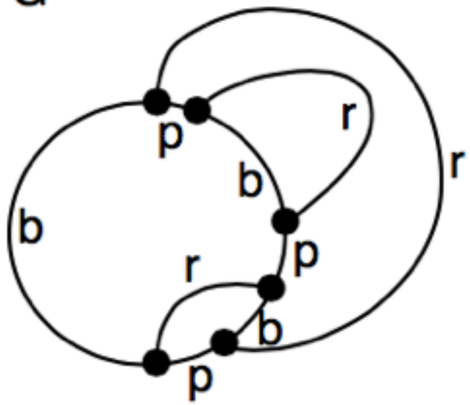


Formation Theorem. Every connected plane cubic graph without isthmus has a formation.

Proof. The statement is equivalent to the FCT.







Coloring of G corresponding to the formation F.

Theorem. The following statement is equivalent to the Map Theorem: Let G be a plane cubic graph with no isthmus. There there exists an even perfect matching of G.

Proof. Let G be a cubic plane graph with no isthmus. Suppose that G is properly 3-colored from the set $\{a, b, c\}$. Let E denote all edges in G that receive the color c. Then, by the definition of proper coloring, the edges in E are disjoint. By the definition of proper 3-coloring every node of G is in some edge of E. Thus E is a perfect matching of G. Since each cycle in $C(E, G)$ is two-colored by the set $\{a, b\}$, each cycle is even. Hence E is an even perfect matching of G.

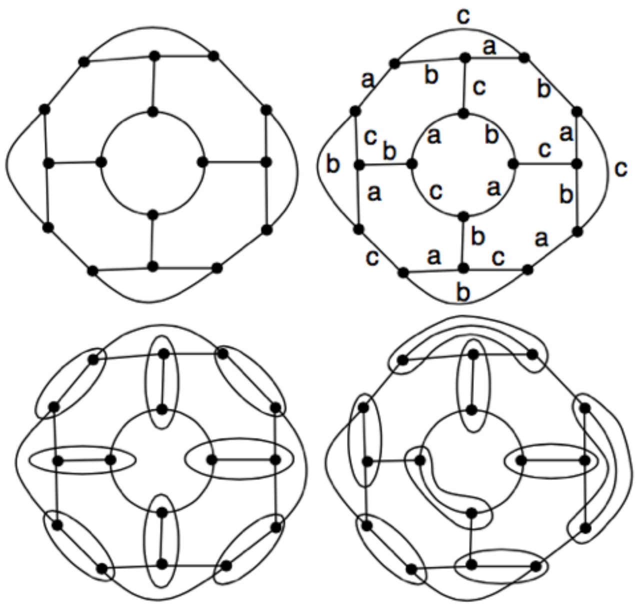


Figure 3: **Perfect Matchings of a Cubic Plane Graph**

Every cubic map with no isthmus has a perfect matching.

D. A. Holton and J. Sheehan, The Petersen graph,
Australian Mathematical Society,
Lecture Notes, vol. 7, Cambridge University Press,
Cambridge, 1993. MR 1232658

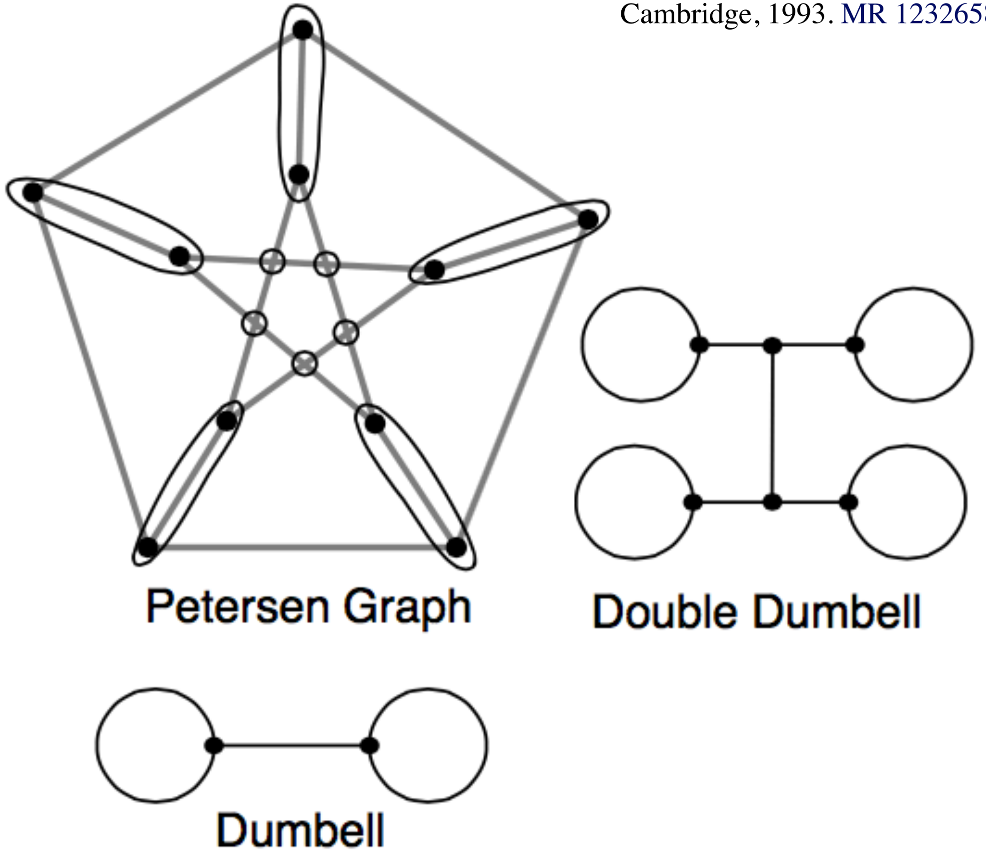
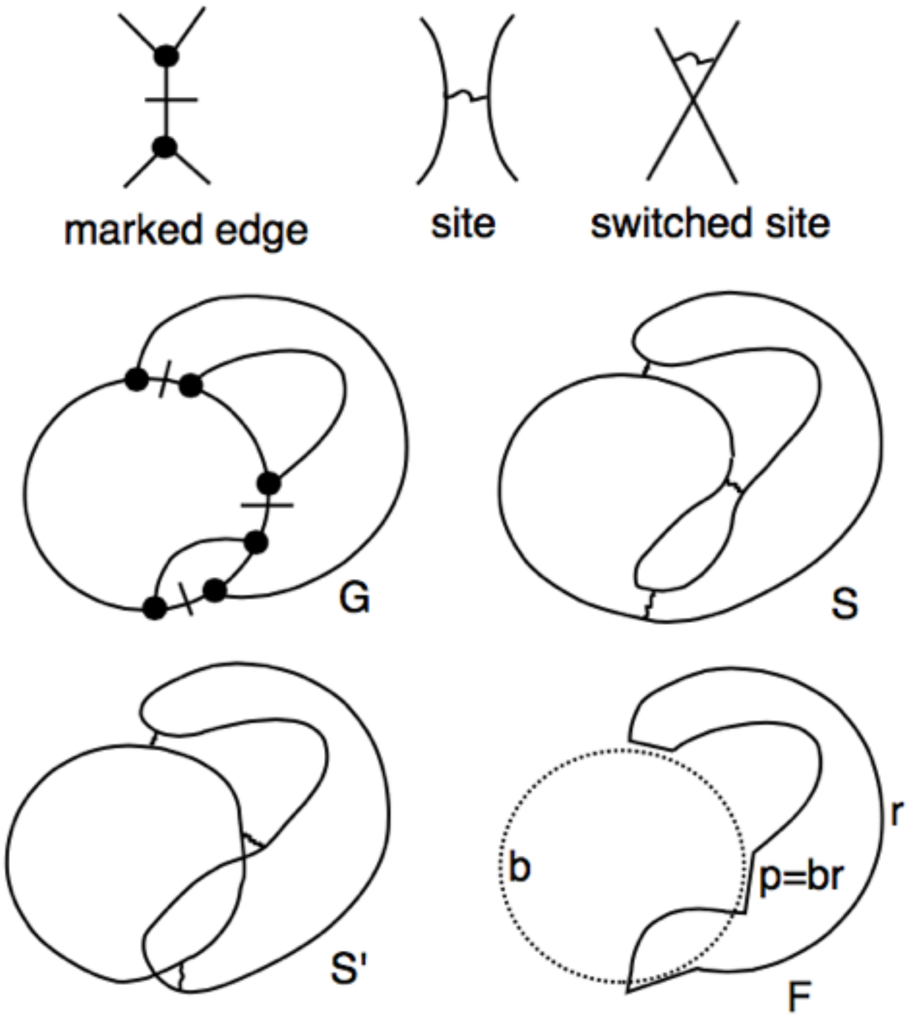
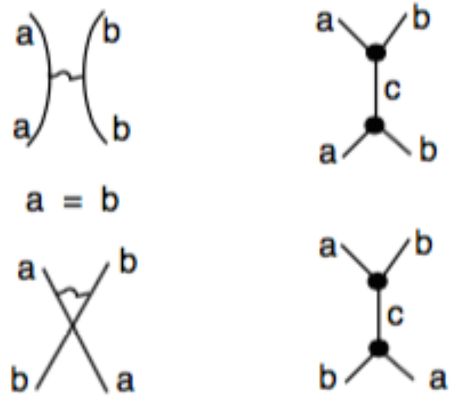


Figure 4: **Petersen and Dumbells**

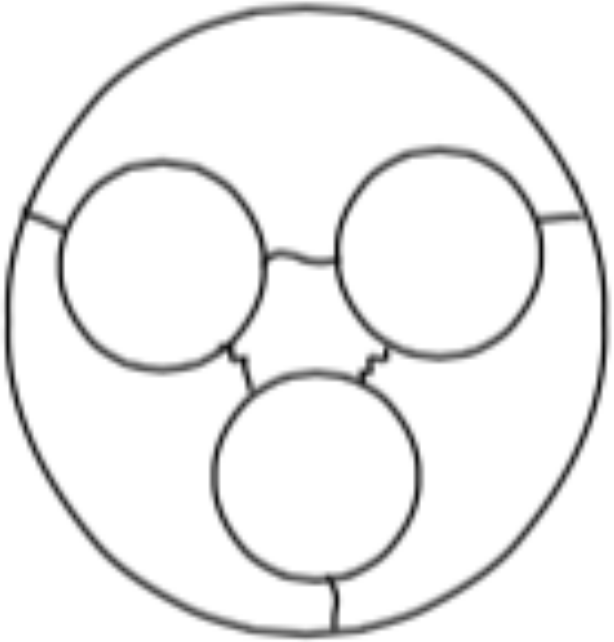
Smoothing and Switching Local States



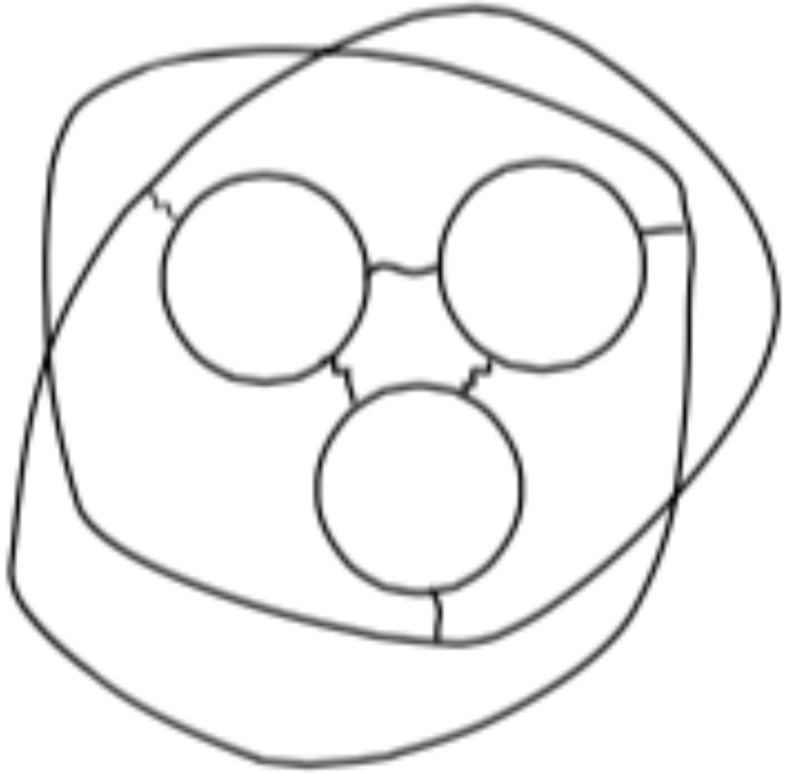


Third Color and Parallel or Permute.

Choose Perfect Matching.
Smooth the sites.
Then SOME switched state is colorable.

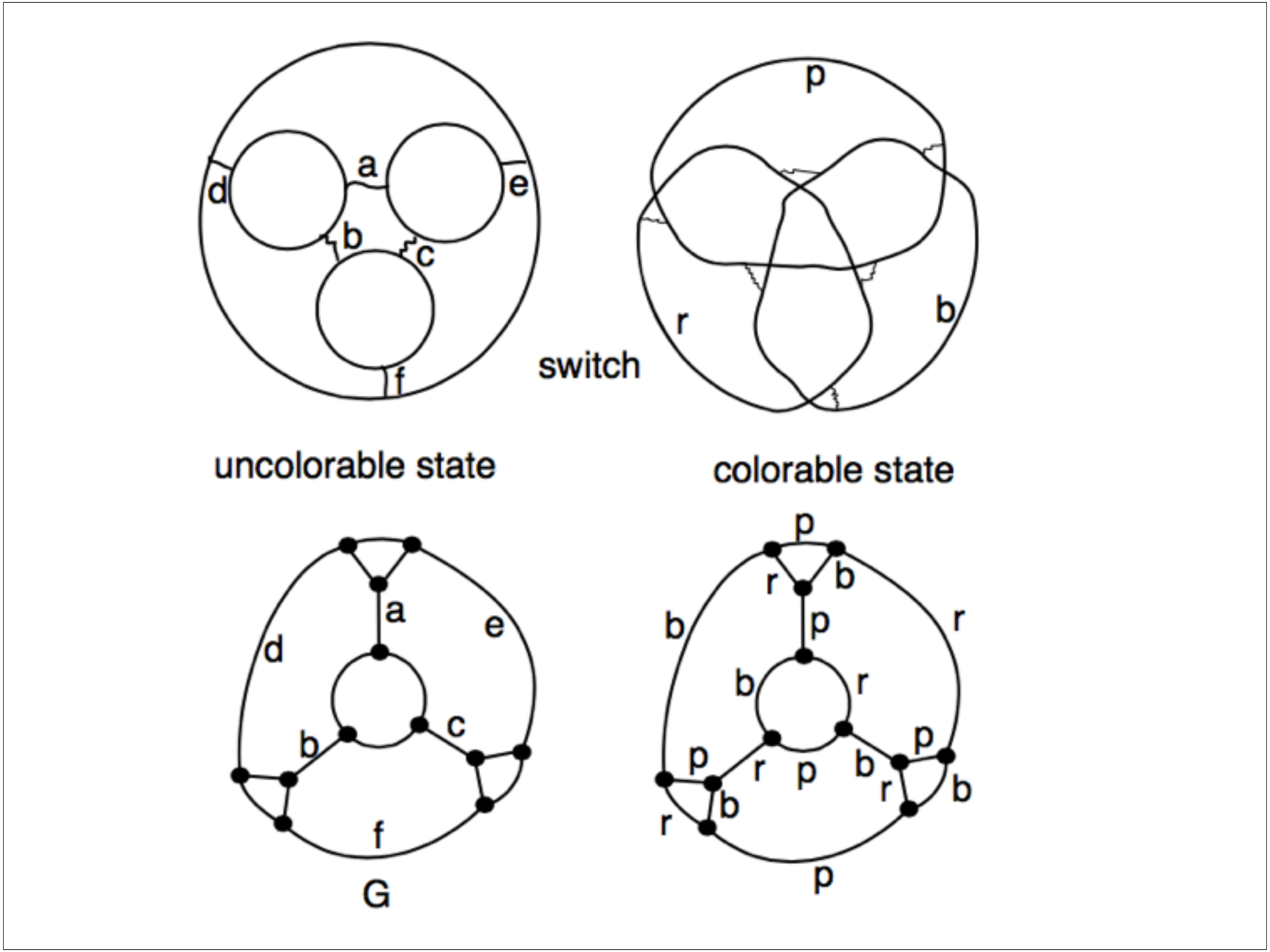


an uncolorable state that is colorable via switching.



a non-planar uncolorable state that cannot be switched to a colorable state.

FCT iff any disjoint collection of Jordan curves in plane with indicated smoothing sites has a colorable switched state.


a
d
e
b
c
f
p
r
b
switch
uncolorable state
colorable state
a
d
e
b
c
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G
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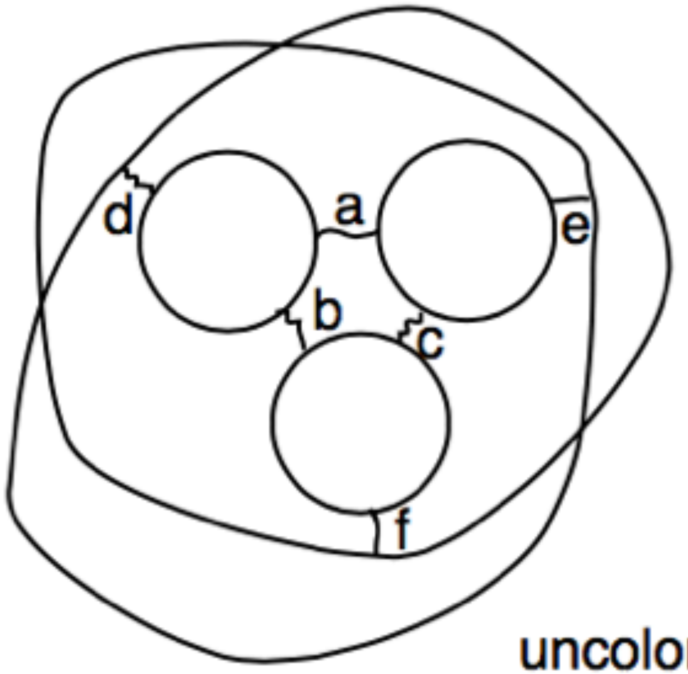
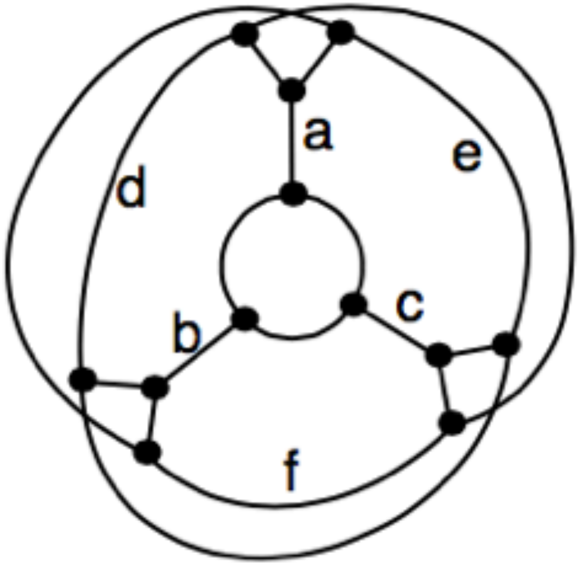
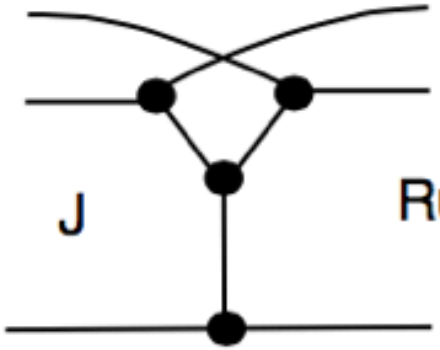


uncolorable



J_3

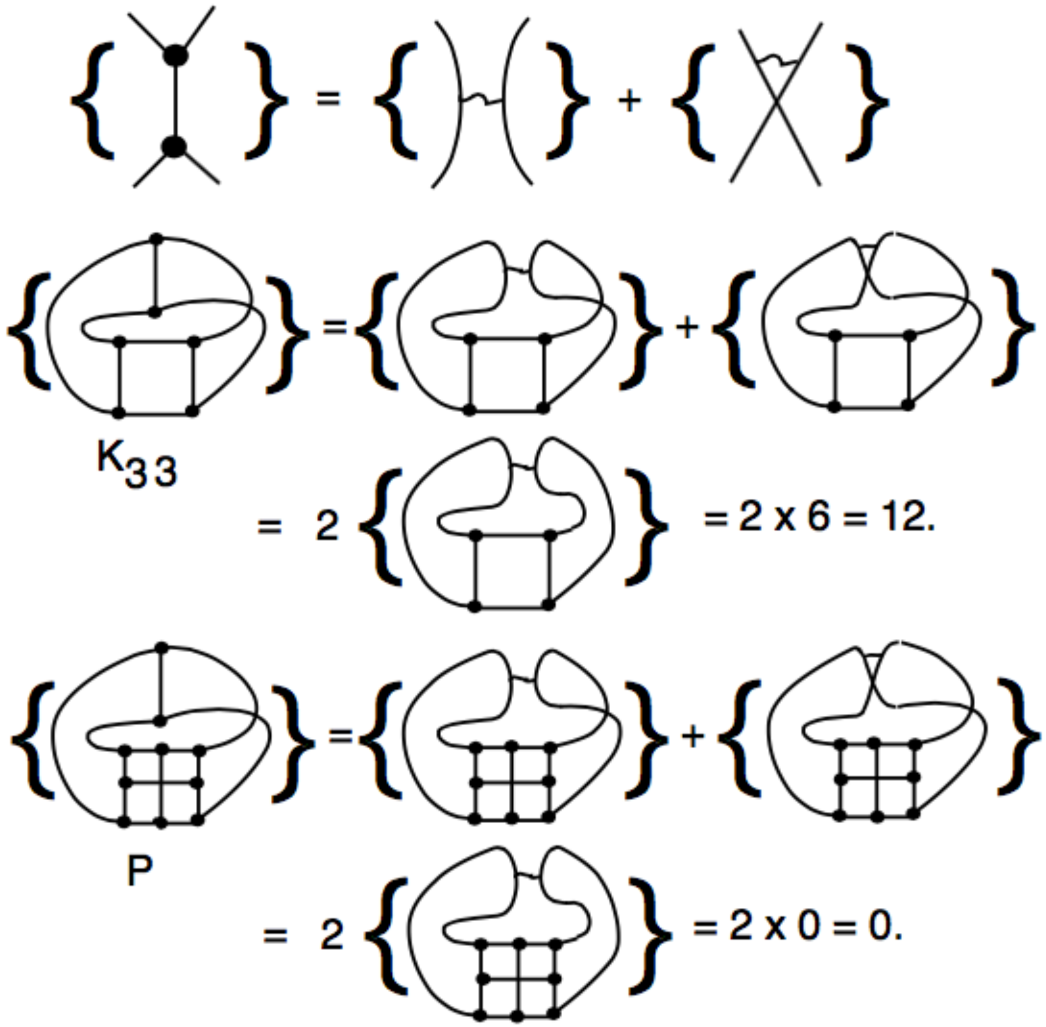


J

Rufus Isaac's J Construction.

A NonPlanar State Corresponds to Isaac's J_3.

A Logical Coloring Expansion, calculating K_33, and showing that Petersen is uncolorable.



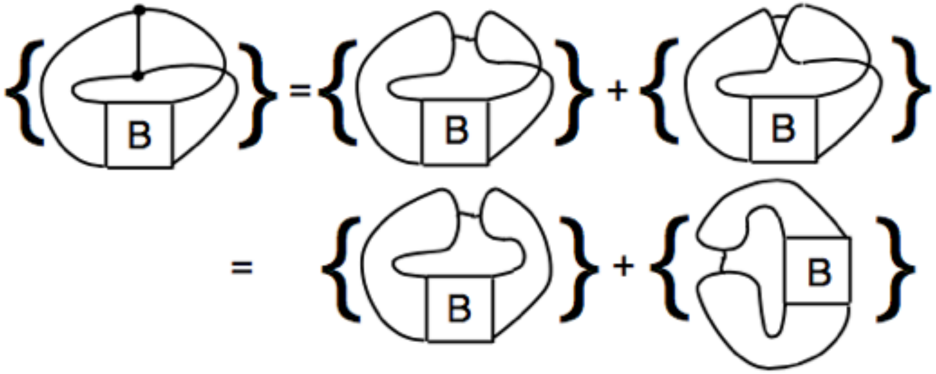


Figure 10: **Expanding a General Edge.**

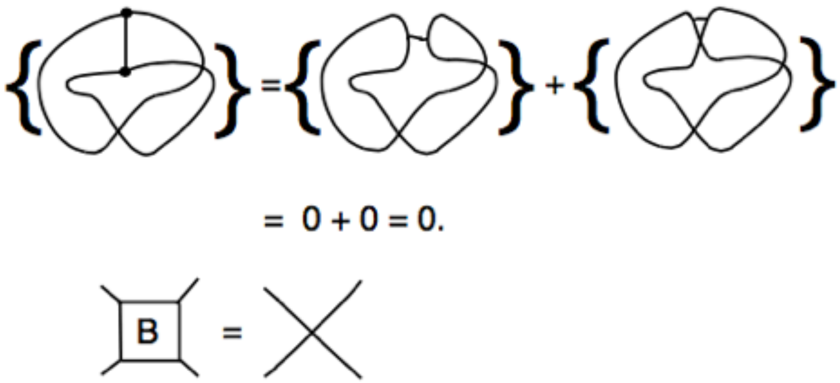
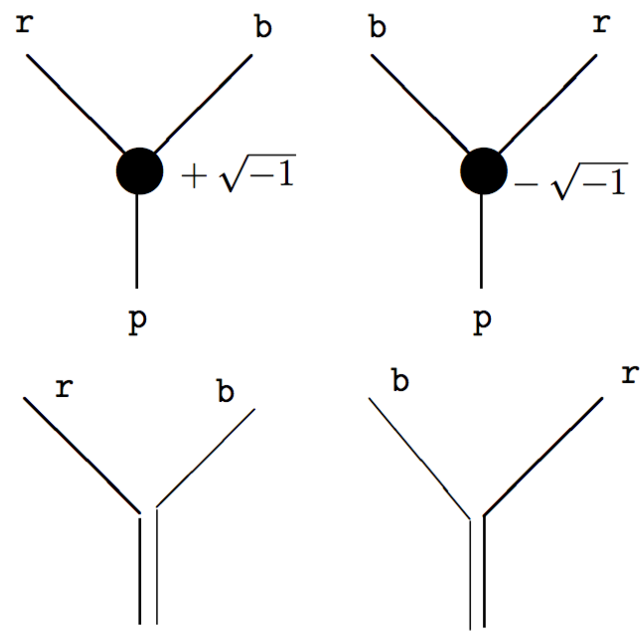


Figure 11: **Dumbell from Crossover Black Box.**

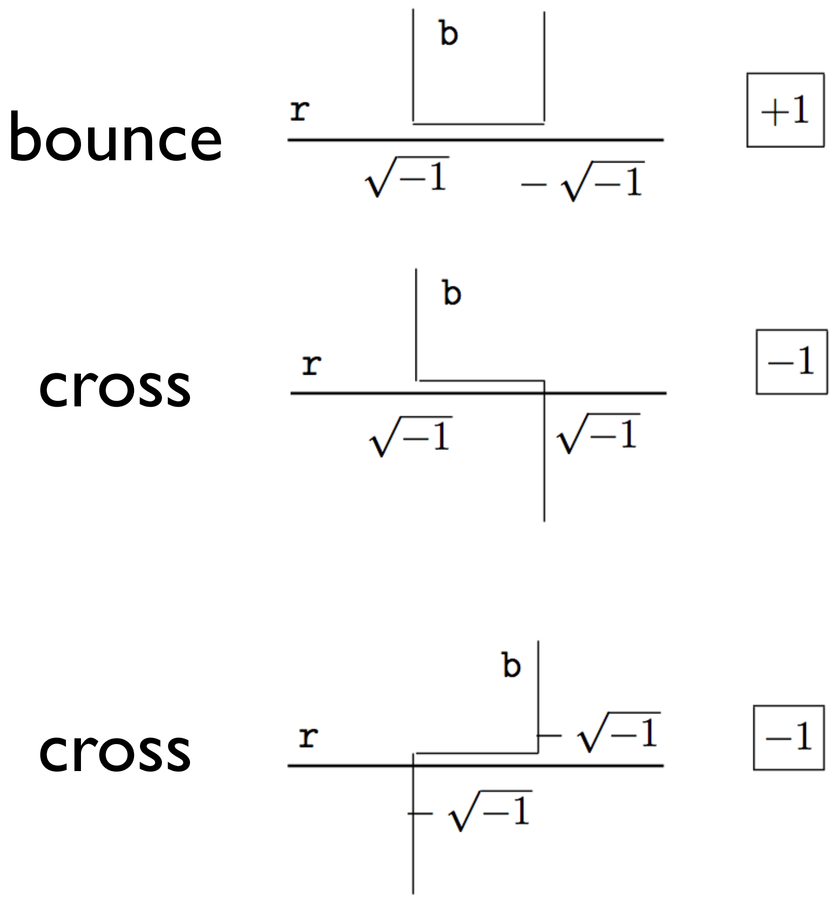
The Penrose Formula

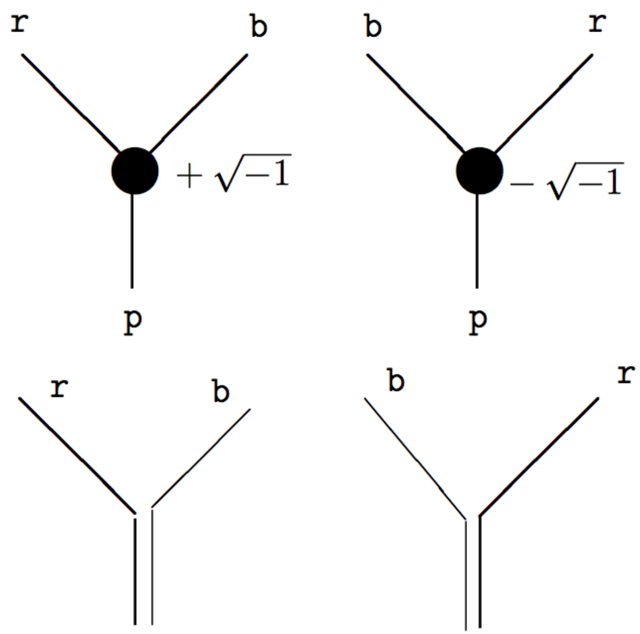
Roger Penrose [6] gives a formula for computing the number of proper edge 3-colorings of a plane cubic graph G. In this formula each vertex is associated with the "epsilon" tensor

$$P_{ijk} = \sqrt{-1}\epsilon_{ijk}$$



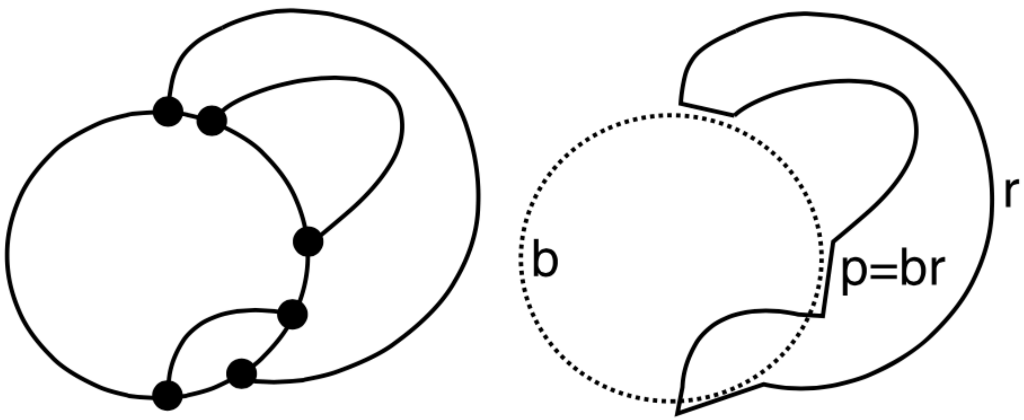
One takes the colors from the set $\{1,2,3\}$ and the tensor ϵ_{ijk} takes value 1 for $ijk = 123, 231, 312$ and -1 for $ijk = 132, 321, 213$. The tensor is 0 when ijk is not a permutation of 123. One then evaluates the graph G by taking the sum over all possible color assigments to its edges of the products of the P_{ijk} associated with its nodes. Call this evaluation $[G]$.



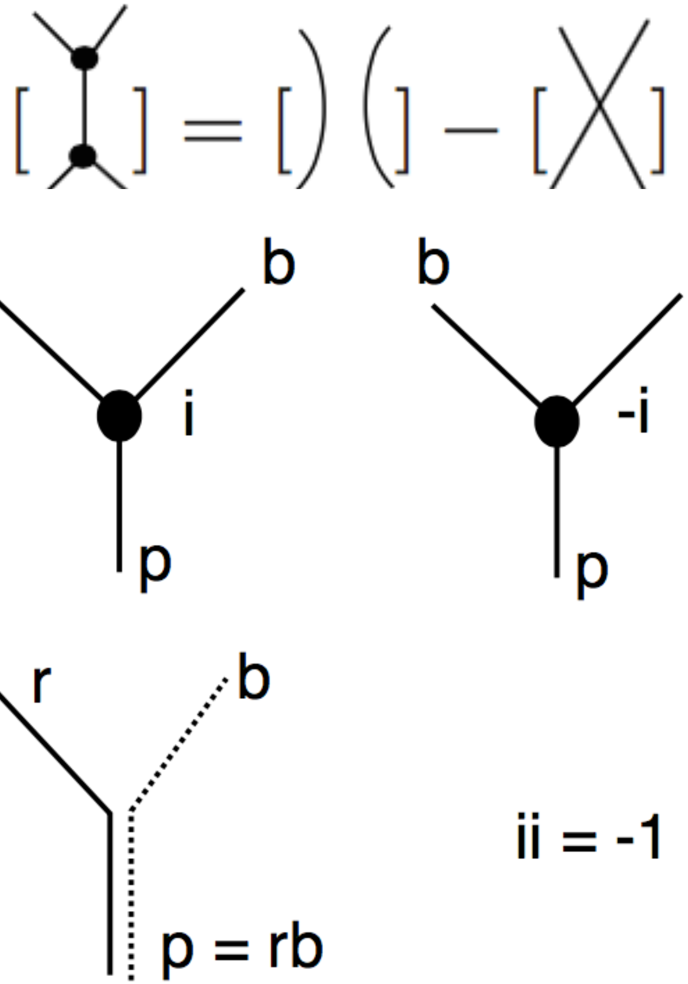


Theorem (Penrose). If G is a planar cubic graph, then $[G]$, as defined above, is equal to the number of distinct proper colorings of the edges of G with three colors (so that every vertex sees three colors at its edges).

Proof. It follows from the above description that only proper colorings of G contribute to the summation $[G]$, and that each such coloring contributes a product of $\pm\sqrt{-1}$ from the tensor evaluations at the nodes of the graph. In order to see that $[G]$ is equal to the number of colorings for a plane graph, one must see that each such contribution is equal to $+1$. The proof of this assertion is given in Figure 21 where we see that in a formation for a coloring each bounce contributes $+1 = -\sqrt{-1}\sqrt{-1}$ while each crossing contributes -1. Since there are an even number of crossings among the curves in the formation, it follows that the total product is equal to $+1$. This completes the proof of the Penrose Theorem.



The Penrose Formula is based on regarding the graph as a contraction of assignments of the epsilon tensor to its nodes.

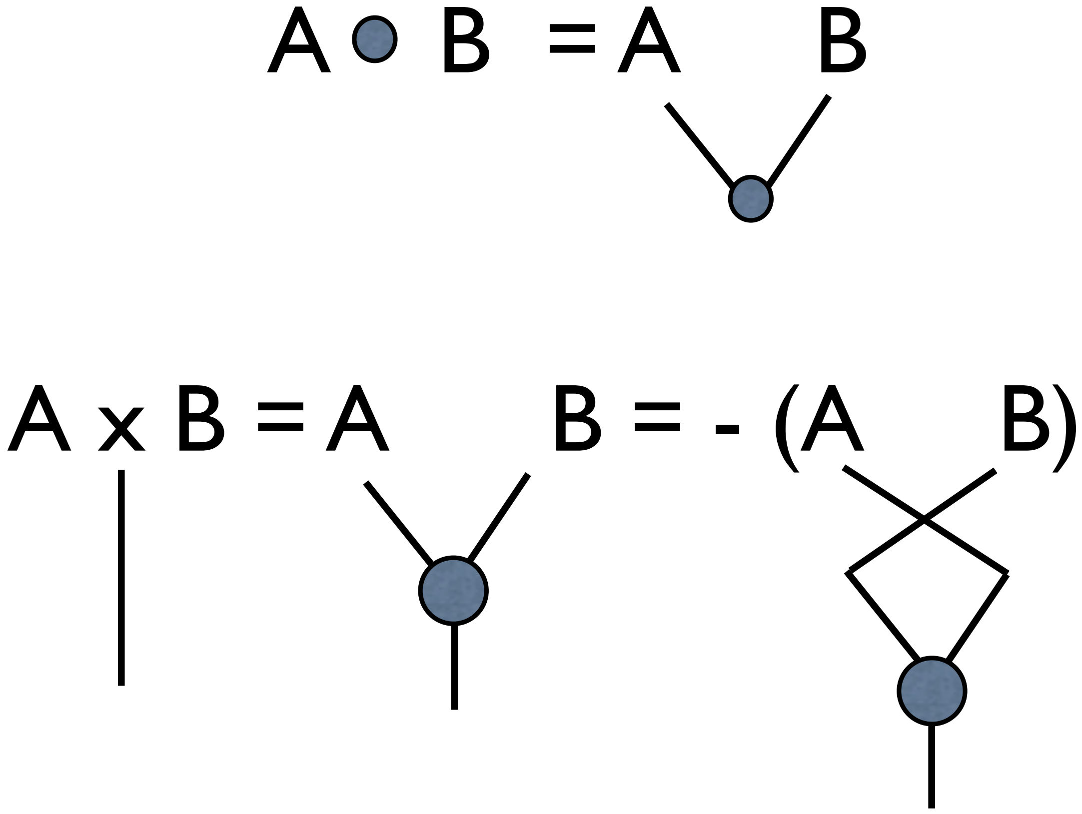


$$[O\,G] = 3[G]$$

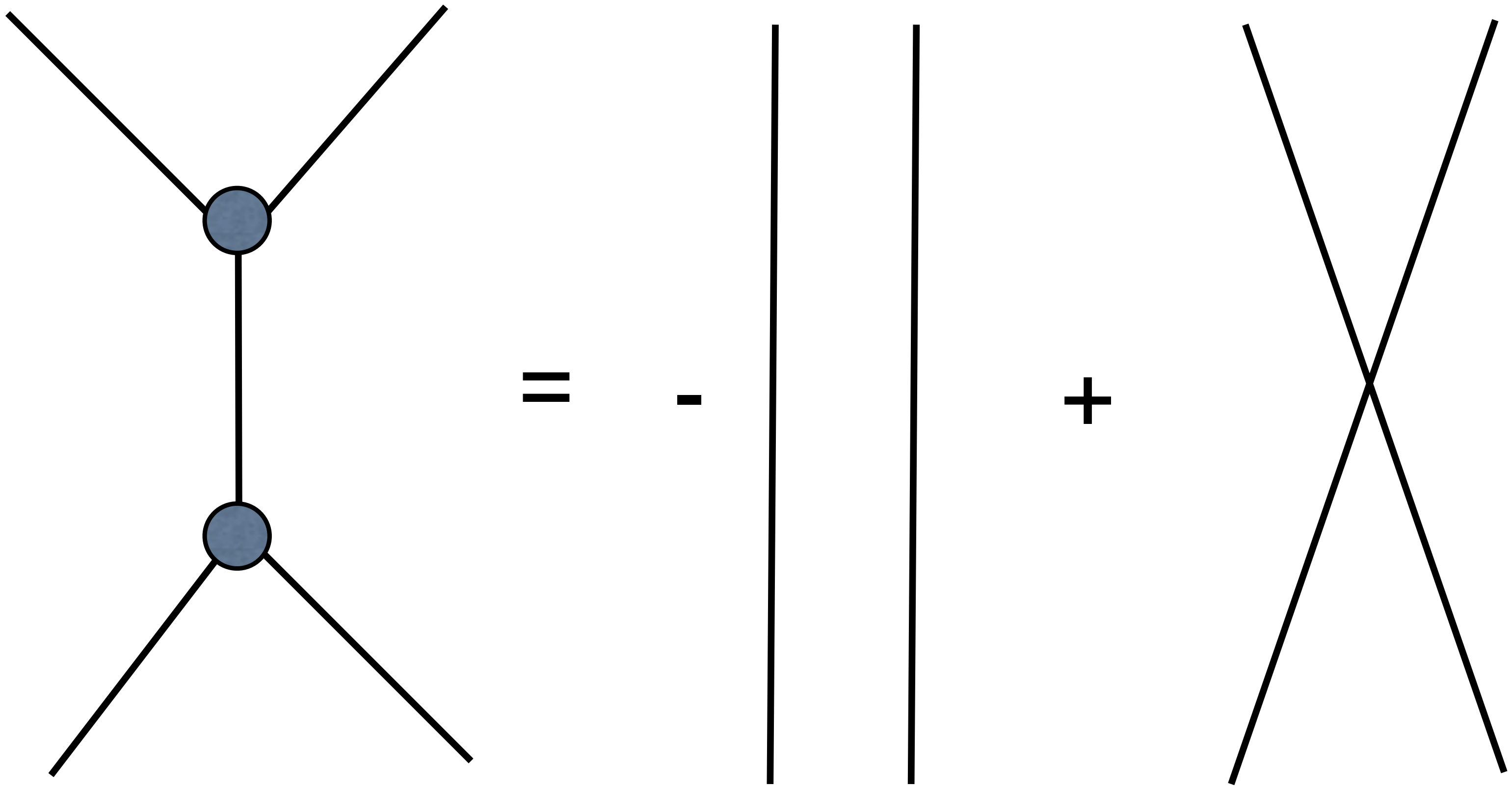
$$[O] = 3.$$

Figure 12: **Node as Epsilon Tensor**

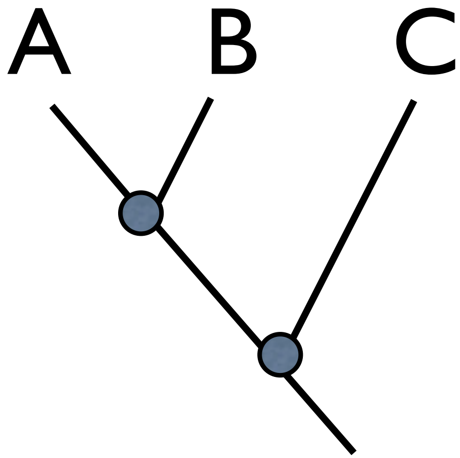
Secrets of Vectors



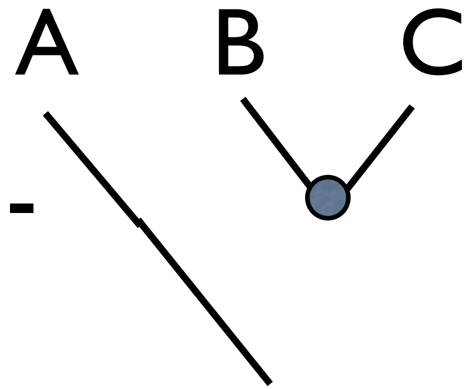
The Epsilon Identity



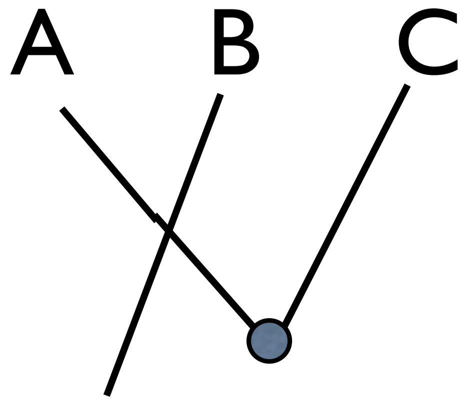
$A \times (B \times C) =$



$=$



$+$



$$= -A\,(B \cdot C) + B(A \cdot C)$$

The Map Theorem is equivalent to the statement that two associated expressions in the vector cross product have evaluations using i,j,k that are non-zero and equal to one another.

For example
$A \times (B \times (C \times D)) = (A \times B) \times (C \times D)$ is
solved by (exercise!).

Because the quaternions are associative, it suffices to find a choice for i,j,k so that the left and right hand sides are both non-zero. Then they will be equal.

The Penrose Formula does not work, as it stands, for non-planar graphs.

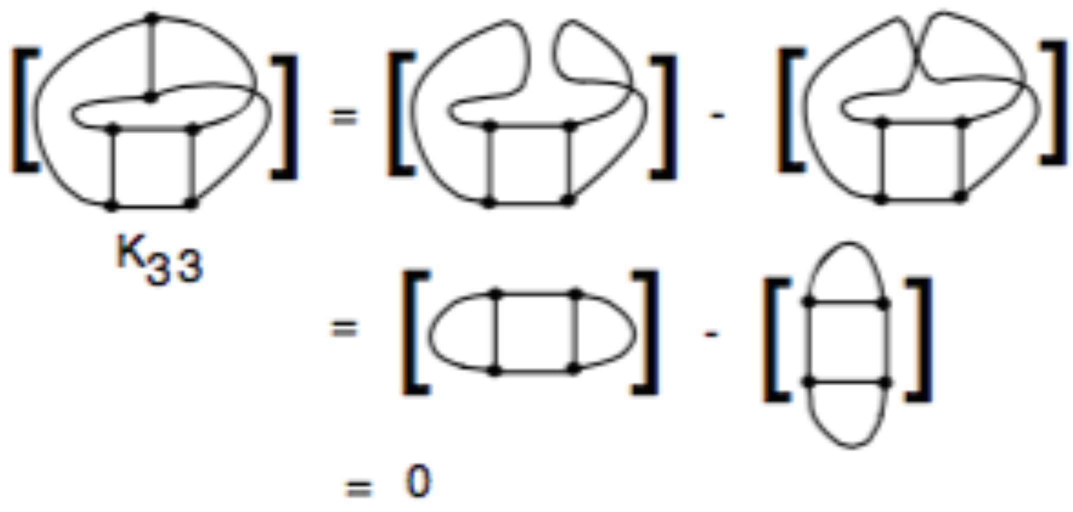
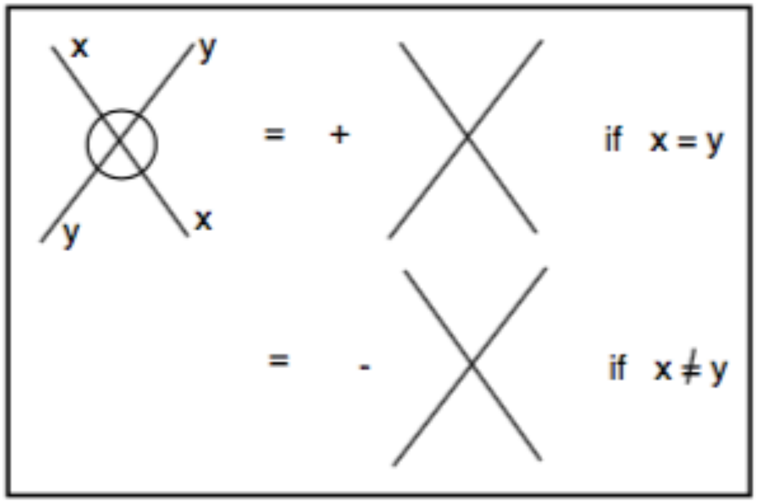
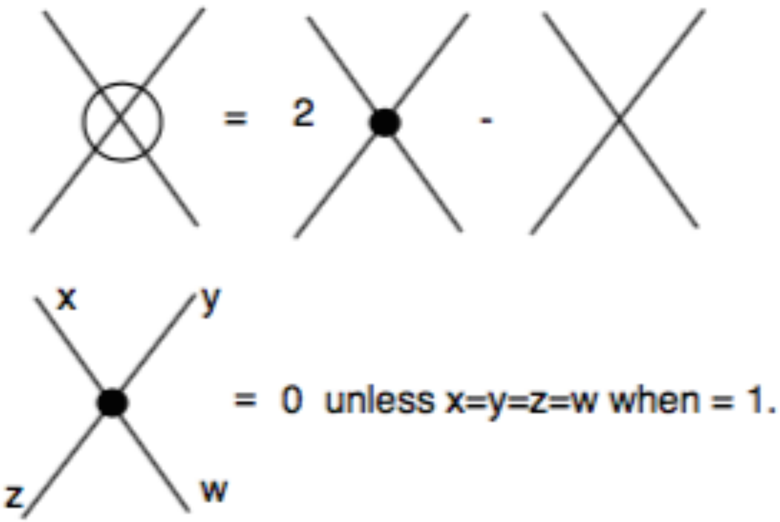


Figure 14: **Penrose on K33 is Zero.**

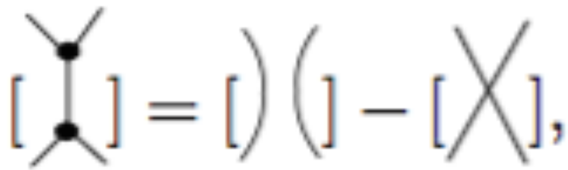
Revising the Penrose Formula for Non-Planar Graphs

For a graph immersed in the plane, with circled immersed crossings, we modify the signs at these virtual crossings when they correspond to crossings of curves of different colors in a formation for the graph.



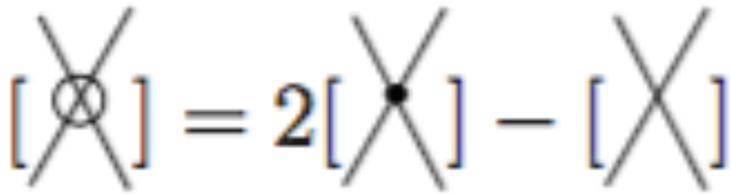


The Revised Penrose Formula



$$[O\,G] = 3[G],$$

$$[O] = 3.$$



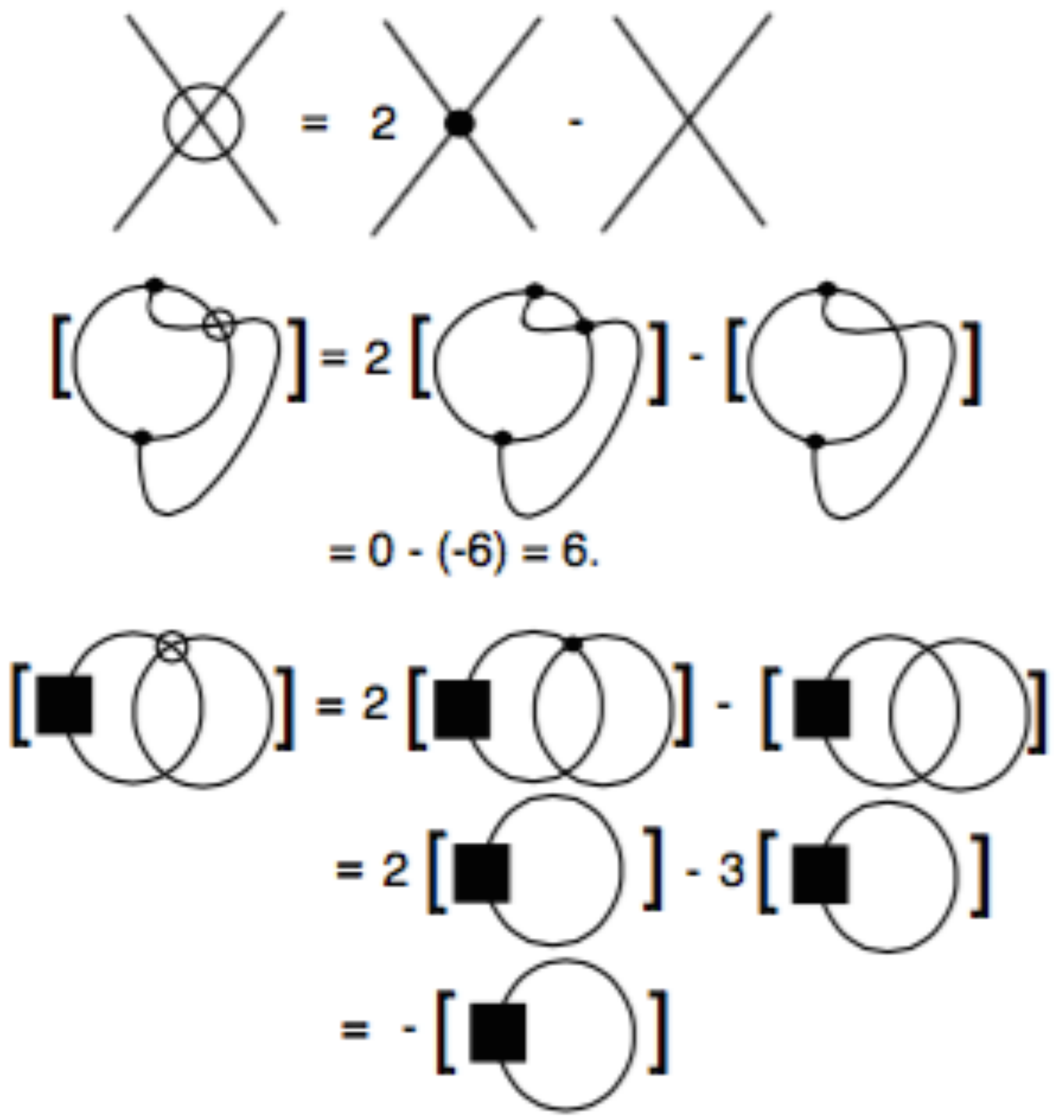


Figure 16: **Crossing Tensor Formalism.**

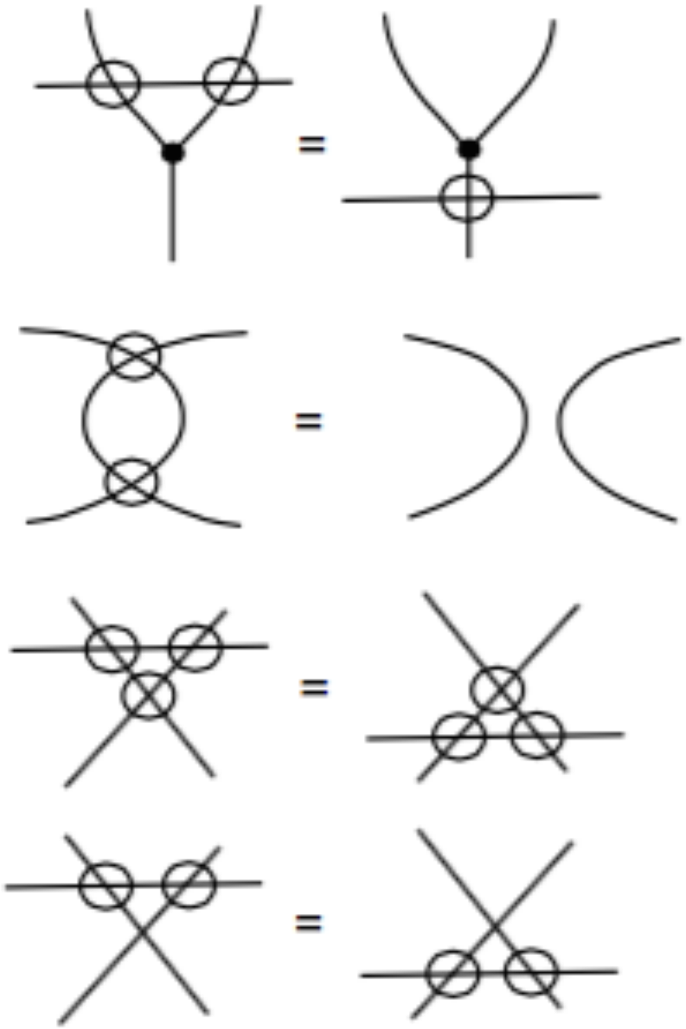
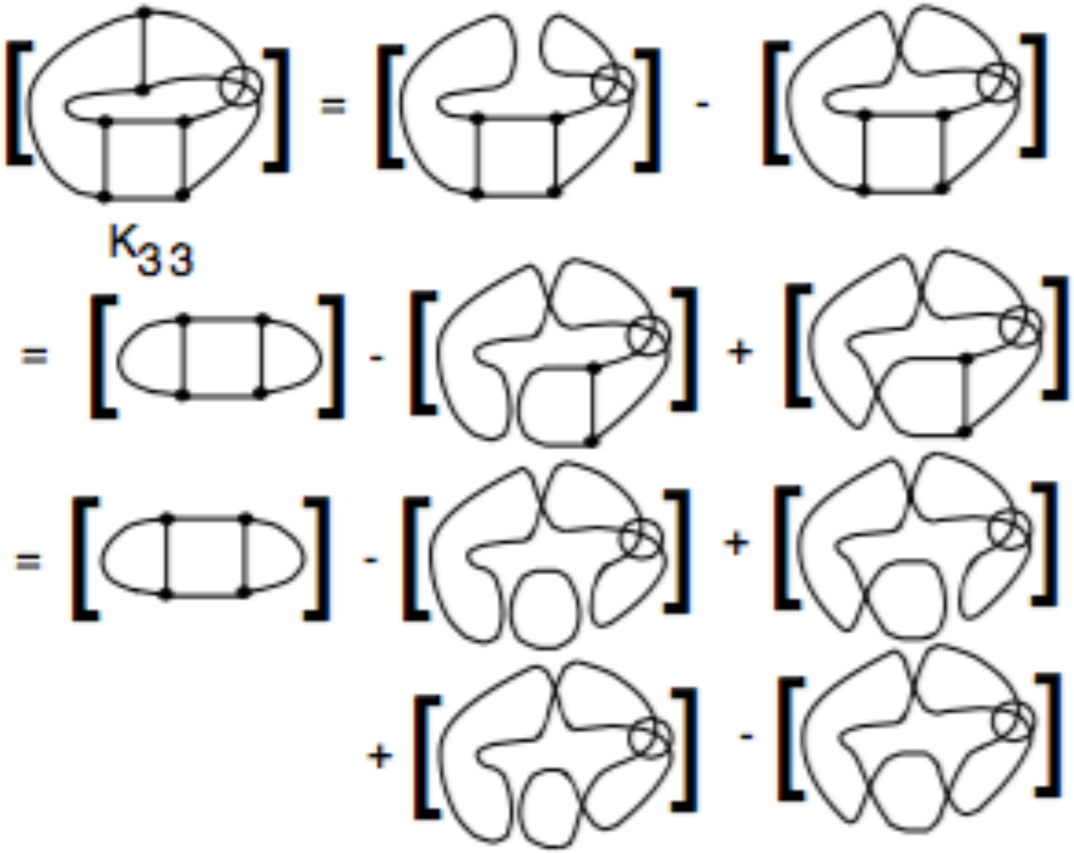


Figure 17: **Topological Tensor Identities.**



$$= \ 12 - 9 + 3 + 3 - (3 - 6) = 12 - 9 + 3 + 3 - 3 + 6 = 12.$$

Figure 18: **Revised Penrose on K33 Counts Colorings.**

Summary

We have shown how smoothing states of a cubic graph are related to edge colorings of the graph with three colors distinct at each node. And we have given two methods to obtain the number of all such colorings of an arbitrary graph.

One method is a logical expansion and leads to structural insight. The other method is a generalization of the Penrose formula for plane graphs.

Both of these methods are related to the combinatorial topology of knots and knot diagrams. This will be the subject of subsequent discussion.

Bracket Polynomial is a Topological Analogue to Penrose Tensor Evaluations

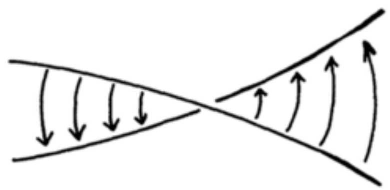
I begin by defining a 3-variable polynomial on unoriented link diagrams. Given an unoriented link diagram K, $[K] \in Z[A, B, d]$ will denote the corresponding polynomial in commuting variables A, B, and d. The bracket polynomial satisfies the *axioms*:

Bracket Axioms

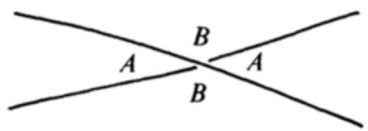
1. $$[\times] = A[\asymp] + B[\,)(\,]$$
   $$[\times] = B[\asymp] + A[\,)(\,]$$

2. $$[O\,K] = d[K]$$
   $$[O] = d.$$

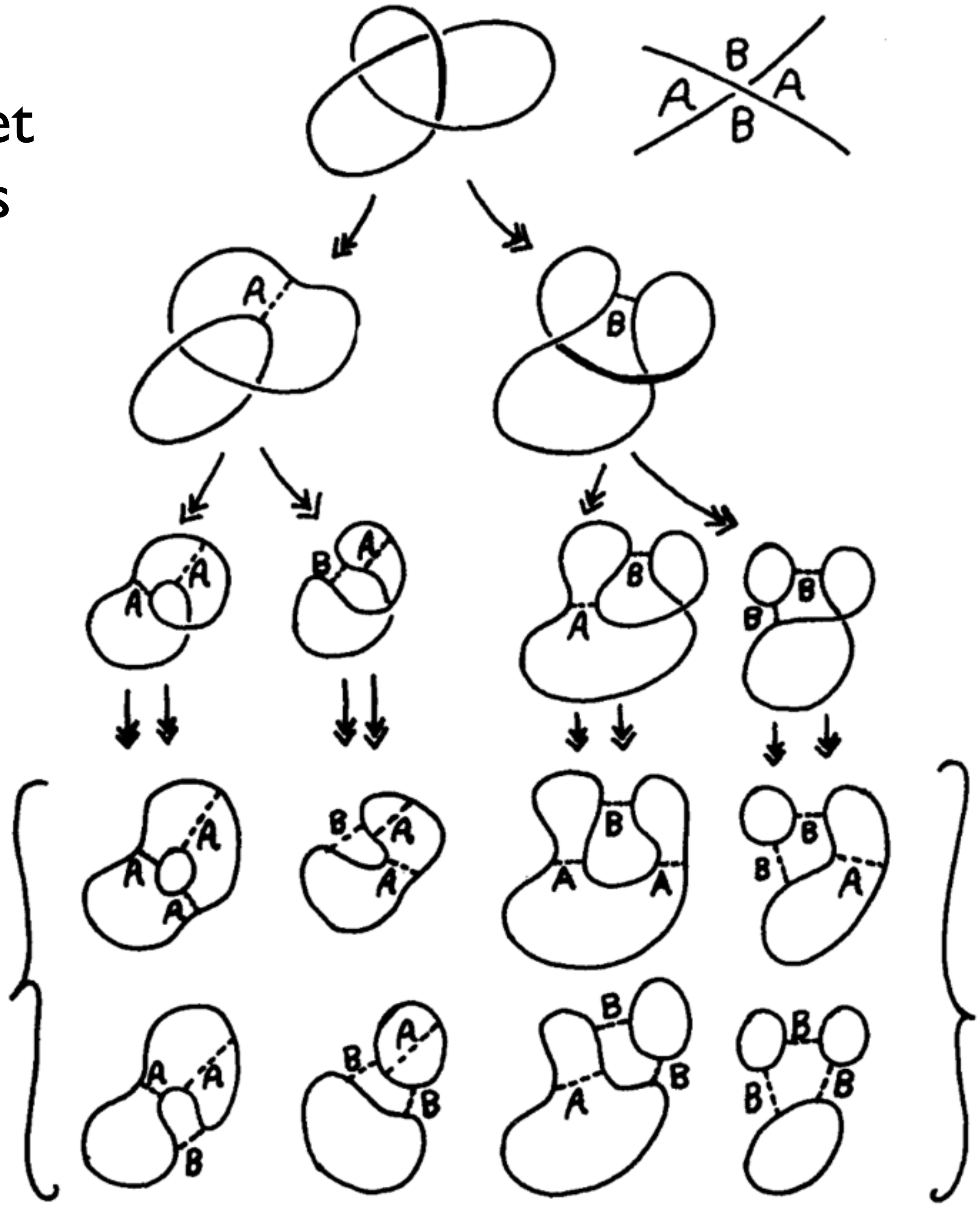
Some explanation of these rules is in order. First note that *an unoriented crossing discriminates two out of the four regions incident at its vertex*. This can be done conventionally by rotating the over-crossing line counterclockwise and choosing the two regions swept out. Thus



By using this convention, we can label the regions A and B, respectively:

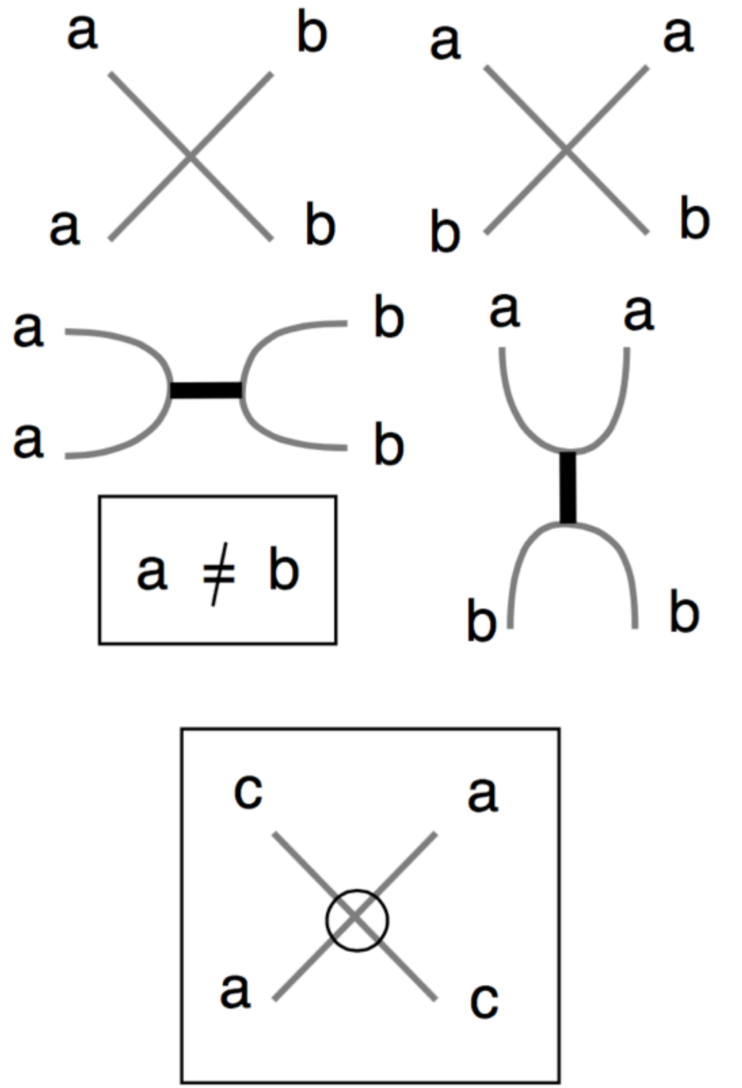


Bracket States

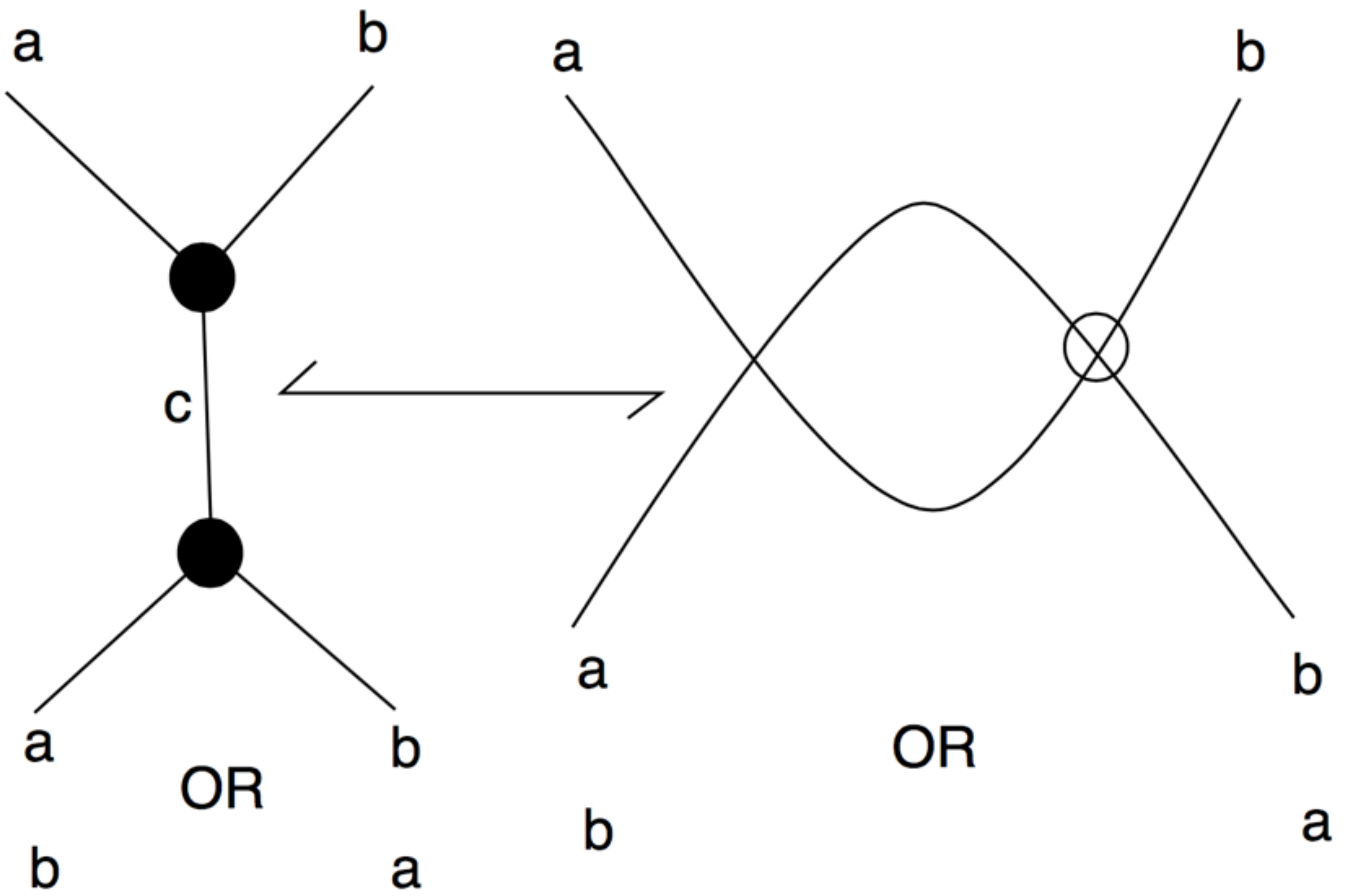


Here we transfer to an exposition of
Virtual Knot Theory.

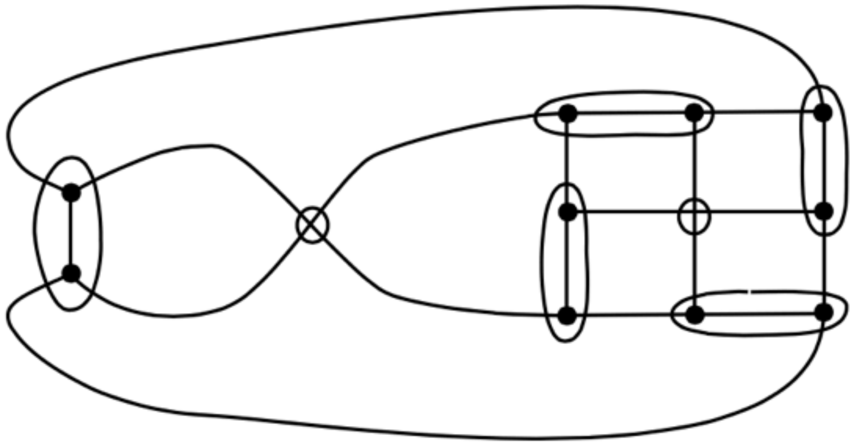
Extend to Coloring Knot and Link Diagrams in a Virtual Context



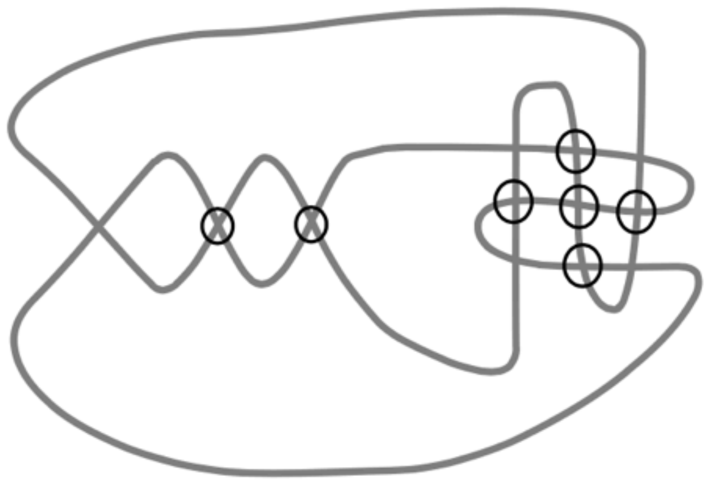
Translation to Virtual Diagrams



Petersen Graph and Corresonding Virtual Shadow Diagram



(G,E)



D(G,E)