Second eigenmatrices of non-commutative association schemes obtained from some designs

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Association Schemes

Let X $(|X| = n)$ be a finite set and $R_i \subset X \times X$ $(i = 0, 1, \ldots, d)$. The adjacency matrix A_i w.r.t. R_i is the square matrix s.t.

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{X} = (X, \{R_i\}_{i=0}^{d})$ is called a *(commutative) association scheme of class* d if the following hold:

(i) $A_0 = I_n$.

(ii) $\sum_{i=0}^{d} A_i = J_n$.

(iii) $\forall i \ \exists i'$ s.t. $A_i^T = A_{i'}$.

(iv) $A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k$ for $\forall i, j$.

(v) $A_i A_j = A_j A_i$ for $\forall i, j$ (commutativity).

Example (The cycle C_4)

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ A symmetric association scheme of class 2.
In general, strongly regular graphs are 2-class symmetric association schemes.

The algebra $\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle_{\mathbb{C}}$ is called the *adjacency algebra of* $\mathcal{X}$.
If $\mathcal{X}$ is commutative

$\Longrightarrow$ (i) $\mathcal{A}$ has $d+1$ primitive idempotents $E_0, E_1, \ldots, E_d$.
(ii) $\{A_0, A_1, \ldots, A_d\}$ and $\{E_0, E_1, \ldots, E_d\}$ are bases of $\mathcal{A}$.
(iii)

$$A_i = \sum_{j=0}^{d} p_i(j) E_j,$$

$$E_i = \frac{1}{n} \sum_{j=0}^{d} q_i(j) A_j.$$

$P = (p_j(i)), Q = (q_j(i))$ are the *first and second eigenmatrices of* $\mathcal{X}$, respectively.

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How about non-commutative case??

Properties of adjacency algebras

Since $\mathcal{A}$ is semisimple, $\mathcal{A} \simeq \bigoplus_{i=0}^{r} \mathrm{M}_{d_i}(\mathbb{C})$ for uniquely determined integers r and $d_1, d_2, \ldots, d_r$.
Then $d+1 = \dim\mathcal{A} = \sum_{i=0}^{r} d_i^2$.
Let φ_i be an irreducible representation from $\mathcal{A}$ to $\mathrm{M}_{d_i}(\mathbb{C})$,
$I_{d_i} \in \mathrm{M}_{d_i}(\mathbb{C})$ be the identity matrix of $\mathrm{M}_{d_i}(\mathbb{C})$ and
$\varepsilon_i^{(j,k)} \in \mathrm{M}_{d_i}(\mathbb{C})$ be the matrix with 1 on (j,k)-entry and 0 otherwise.

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Then following hold:

- $\exists E_0, E_1, \ldots, E_r \in \mathcal{A}$ s.t. $\varphi_i(E_j) = \delta_{i,j} I_{d_j}$.
  These are uniquely determined and called *central idempotents*.
- Let $E_i^{(j,k)} \in \mathcal{A}$ s.t. $\varphi_l(E_i^{(j,k)}) = \delta_{l,i}\varepsilon_i^{(j,k)}$.
  Then $\{E_i^{(j,k)} \mid 0 \le i \le r, 1 \le j,k \le d_i\}$ is a basis of $\mathcal{A}$.

Since choices of $E_i^{(j,k)}$ depend on φ_i, $E_i^{(j,k)}$ are NOT uniquely determined. Note that $E_i = \sum_{j=1}^{d_i} E_i^{(j,j)}$.

How to determine $E_i^{(j,k)}$??

If $\mathcal{A}$ is non-commutative, then $d_i \geq 2$ for some i.

Assume $d_i \in \{1, 2\}$ for all i and $\mathcal{A}$ has a subalgebra $\mathcal{A}'$ satisfying following conditions:

- $\mathcal{A}'$ is an adjacency algebra of a commutative (fusion) association scheme.
- Let $E'_0, E'_1, \ldots, E'_s$ be the primitive idempotents of $\mathcal{A}'$.
  If $d_i = 2 \Longrightarrow \exists j$ s.t. $E_i E'_j = E'_j$.(*)

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In this assumption, we determine $E_i^{(j,k)}$ as follows:

(i) If $d_i = 1 \Longrightarrow E_i^{(1,1)} = E_i$,

(ii) If $d_i = 2 \Longrightarrow E_i^{(1,1)} = E'_j$, where E'_j satisfies (*),

$$E_i^{(2,2)} = E_i - E_i^{(1,1)},$$
$$E_i^{(1,2)} = E_i^{(1,1)} A_j E_i^{(2,2)} (\neq 0) \text{ for some } A_j,$$
$$E_i^{(2,1)} = E_i^{(2,2)} A_j E_i^{(1,1)} (\neq 0) \text{ for some } A_j.$$

Second eigenmatrices

Since $E_0, \ldots, E_r$ and $E'_0, \ldots, E'_s$ are expressed as a sum of A_l, we can write $E_i^{(j,k)}$ as a sum of A_l:

$$E_i^{(j,k)} = \sum_{l=0}^{d} q_i^{(j,k)}(l) A_l.$$

The second eigenmatrix of $\mathcal{A}$ is a square matrix

$$Q = (q_i^{(j,k)}(l)).$$

Quasi-symmetric designs

Definition

Let $\mathcal{P}$ be a point set with $|\mathcal{P}| = v$ and $\mathcal{B} \subset \binom{\mathcal{P}}{k}$ for $1 \le k \le v$. A pair $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is called *a* t-(v,k,λ)*design* if $\lambda = \#\{B \in \mathcal{B} \mid T \subset B\}$ for any $T \in \binom{\mathcal{P}}{t}$.
A t-(v,k,λ)design is *quasi-symmetric* if $\#\{\#(B_1 \cap B_2) \mid B_1, B_2 \in \mathcal{B}, B_1 \ne B_2\} = 2$.

Example

$\mathcal{P} = \{1,2,3,4,5,6\}$.
$\mathcal{B} = \{\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,6\},\{1,5,6\},$
$\{2,3,6\},\{2,4,5\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}$.
$\Longrightarrow$ quasi-symmetric 2-$(6,3,2)$design with intersection numbers $\{1,2\}$.

Flags of quasi-symmetric designs

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a quasi-symmetric 2-$(v, k, 1)$design. In this case, the set of intersection numbers is $\{0, 1\}$.
Let $\mathcal{F} = \{(p, B) \in \mathcal{P} \times \mathcal{B} \mid p \in B\}$ be the set of flags and $R_0, R_1, \ldots, R_6$ be relations on $\mathcal{F}$ defined as

$$
\begin{aligned}
R_0 &= \{(f, f) \mid f \in \mathcal{F}\},\\
R_1 &= \{(f, g) \mid p \neq q, B = C\},\\
R_2 &= \{(f, g) \mid \#(B \cap C) = 0\},\\
R_3 &= \{(f, g) \mid p = q, \#(B \cap C) = 1\},\\
R_4 &= \{(f, g) \mid p \neq q, \#(B \cap C) = 1, q \notin B, p \in C\},\\
R_5 &= \{(f, g) \mid p \neq q, \#(B \cap C) = 1, p \in B, p \notin C\},\\
R_6 &= \{(f, g) \mid p \neq q, \#(B \cap C) = 1, q \notin B, p \notin C\},
\end{aligned}
$$

where $f = (p, B), g = (q, C) \in \mathcal{F}$.

Non-commutative association schemes obtained from flags

Theorem (Klin-Munemasa-Muzychuk-Zieschang(2003))

For any quasi-symmetric 2-$(v,k,1)$*design,* $(\mathcal{F},\{R_i\}_{i=0}^{6})$ *is a non-commutative association scheme.*

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Since block graphs of quasi-symmetric designs are strongly regular graphs,

$$\{R_0\cup R_1, R_2, R_3\cup R_4\cup R_5\cup R_6\}$$

gives a quotient association scheme.
Thus

$$\{R_0, R_1, R_2, R_3\cup R_4\cup R_5\cup R_6\}$$

gives a fusion association scheme.

Second eigenmatrices of flag algebras

Let A_i be the adjacency matrix w.r.t. R_i and

$$\mathcal{A} = \langle A_0, A_1, \ldots, A_6 \rangle_{\mathbb{C}}.$$

The algebra $\mathcal{A}$ has 4-dim. subargebra

$$\mathcal{A}' = \langle A_0, A_1, A_2, A_3 + A_4 + A_5 + A_6 \rangle_{\mathbb{C}} \subset \mathcal{A}$$

Property of $\mathcal{A}$ and $\mathcal{A}'$

- $\mathcal{A} \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C})$.
  $\Longrightarrow$ central idempotents E_0, E_1, E_2, E_3.
- $\mathcal{A}' \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ ($\mathcal{A}'$ is a commutative algebra.)
  $\Longrightarrow$ primitive idempotents E'_0, E'_1, E'_2, E'_3
  with $E_0 = E'_0, E_1 = E'_1, E_3 E'_3 = E'_3, (E_3 - E'_3) + E_2 = E'_2$.

The main result

Theorem

For any quasi-symmetric $2\text{-}(v,k,1)$*design,*

$$\{E_0, E_1, E_2, E_3^{(1,1)}, E_3^{(2,2)}, E_3^{(1,2)}, E_3^{(2,1)}\}$$

is a second basis of $\mathcal{A}$*, where*

$$\begin{aligned}
E_3^{(1,1)} &= E_3',\\
E_3^{(2,2)} &= E_3 - E_3',\\
E_3^{(1,2)} &= E_3^{(1,1)} A_3 E_3^{(2,2)},\\
E_3^{(1,2)} &= E_3^{(2,2)} A_3 E_3^{(1,1)}.
\end{aligned}$$

Moreover we can compute the second eigenmatrix Q *of* $\mathcal{A}$*.*

Hadamard 3-designs

Hadamard 3-designs are quasi-symmetric 3-$(4n, 2n, n-1)$designs with intersection numbers $\{0, n\}$ $(n \geq 2)$.
Let $\mathcal{F}$ be the set of flags of a Hadamard 3-design. We define relations

$$
\begin{aligned}
R_0 &= \{(f,f) \mid f \in \mathcal{F}\},\\
R_1 &= \{(f,g) \mid p \neq q, B = C\},\\
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R_7 &= \{(f,g) \mid p \neq q, \#(B \cap C) = n, q \in B, p \in C\}.
\end{aligned}
$$

Hadamard 3-designs(cont.)

The adjacency algebra $\mathcal{A}$ of $(\mathcal{F}, \{R_i\}_{i=0}^{7})$ has a 4-dim. subalgebra

$$\mathcal{A}' = \langle A_0, A_1, A_2, A_3 + A_4 + A_5 + A_6 + A_7 \rangle_{\mathbb{C}} \subset \mathcal{A}.$$

Moreover

$$\begin{aligned} \mathcal{A} &\simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C}), \\ \mathcal{A}' &\simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}. \end{aligned}$$

$\Longrightarrow$ We can compute the second eigenmatrix.

Generalized quadrangles

A generalized quadrangle is an incidence structure $(\mathcal{P}, \mathcal{B})$ satisfying certain properties.
Let $\mathcal{F}$ be the set of a flags of a generalized quadrangle.
Klin-Pech-Zieschang(1998) proved that $(\mathcal{F}, \{R_i\}_{i=0}^{7})$ is a non-commutative association scheme.
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Thank you for your attention!!