Recent results on Majorana representations of the symmetric groups

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Majorana representations have been introduced by Ivanov around 2009 to provide an abstract framework for studying the actions of the Monster group M and (some of) its subgroups on the Conway-Greiss-Norton algebra (or Monster algebra) V_M, a 196884-dimensional Majorana algebra over $\mathbb{R}$.

One of the advantages of Ivanov's approach is that, starting from rather elementary axioms, it leads to deep insights into the Monster algebra and the action of M on it.

This is particularly important for the later generation of finite simple group theorists (like mine) who came into play too late to follow up the making of the theory and have often to cope with concepts that are hard to understand if you missed their origin.

In this talk I shall

1. define Majorana algebras as a special type of Frobenius axial algebras;
2. define Majorana representations;
3. briefly expose the state of art about Majorana representations;
4. expose and justify some methods taken from algebraic combinatorics.

1. Axial, Frobenius and Majorana algebras

Axial algebras are a generalisation of commutative associative algebras, where associativity is replaced with a fusion rule.

If e is an idempotent element of a commutative associative algebra A, then, for the action $ad(e)$ of e by multiplication, A splits as the direct sum of the two eigenspaces A_1 and A_0 relative to the eigenvalues 1 and 0 .

Moreover, with respect to the algebra product, we have

$$A_1 A_1 \subseteq A_1 \quad A_1 A_0 \subseteq A_0 \quad \text{and } A_0 A_0 \subseteq A_0.$$

that is, the eigenvalues of $ad(e)$ satisfy the following *fusion rule*

$\star$	1	0
1	1	0
0	0	0

Given a field k and a finite subset $\mathcal{S}$ of k, a **fusion rule** on $\mathcal{S}$ is a map

$$\star\colon \mathcal{S} \times \mathcal{S} \to 2^{\mathcal{S}}.$$

An **axial algebra** with **spectrum** $\mathcal{S}$ and fusion rule $\star$ is a commutative non-associative algebra V generated by a set of idempotent elements (called **axes**) such that, for each axis a,

Ax1 $ad(a)$ is a semisimple endomorphism of V with spectrum $\mathcal{S}$;

Ax2 for every $\lambda, \mu \in \mathcal{S}$, $V_\lambda V_\mu \leq \oplus_{\delta \in \lambda \star \mu} V_\delta$

Further V is called **primitive** if,

Ax3 $V_1 = \langle a \rangle$.

Note that, apart from trivial cases, $\mathcal{S}$ has to contain 0 and 1.

A **Frobenius axial algebra** is a pair (V, σ) where V is an axial algebra and $\sigma\colon V \times V \to k$ is a scalar product on V that **associates** with the algebra product, i.e. for every $u, v, w \in V$

Ax4 $\sigma(uv, w) = \sigma(u, vw)$.

The main properties of associativity for symmetric bilinear forms are

PF1 the radical is an ideal of the algebra

PF2 for each axis a, eigenspaces for distinct eigenvalues of $ad(a)$ are orthogonal to each other.

Majorana algebras are primitive real Frobenius axial algebras such that axes have lenght 1 and satisfy the following fusion rule

$\star$	1	0	$1/4$	$1/32$
1	1	0	$1/4$	$1/32$
0	0	0	$1/4$	$1/32$
$1/4$	$1/4$	$1/4$	$\{1, 0\}$	$1/32$
$1/32$	$1/32$	$1/32$	$1/32$	$\{1, 0, 1/4\}$

Surprising as it might be at a first glance, these conditions allow to compute effectively within Majorana algebras.

Indeed there is a complete classification and description of **dihedral** Majorana algebras, i.e. Majorana algebras generated by two axes, which is the starting point for Majorana representations.

This result was originally proven by Norton for V_M, by Sakuma in the context of V.O.A's, and by Ivanov, Pasechnik, Seress and Shpectorov within the axial algebras axiomatics (and further generalised by J. Hall, Shpectorov and Rehren) and goes under the name of Norton-Sakuma Theorem.

By the Norton-Sakuma Theorem, dihedral algebras fall into 9 isomorphism types which are labelled

$$1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, \text{ and } 3C,$$

according to the conjugacy class of the product in M of certain involutions associated to the axes which I'll soon define.

In particular Norton-Sakuma Theorem gives for each of these dihedral algebras

1. a basis;
2. the corresponding structure constants;
3. the corresponding Gram matrix.

As vector spaces over $\mathbb{R}$, dihedral algebras are spanned by axes except for those of type $3A, 4A, 5A$, and $6A$, where, together with the axes, a further vector, called $3-$, $4-$, $5-$, and $3-$axis, respectively, is needed.

Majorana representations

Looking back at the fusion rule for Majorana Algebras,

$\star$	1	0	$1/4$	$1/32$
1	1	0	$1/4$	$1/32$
0	0	0	$1/4$	$1/32$
$1/4$	$1/4$	$1/4$	$\{1,0\}$	$1/32$
$1/32$	$1/32$	$1/32$	$1/32$	$\{1,0,1/4\}$

one sees easily that, for each axis a, the map that inverts every element of its $1/32$-eigenspace and fixes each element of the other eigenspaces is an involutory automorphism t_a of the algebra and, since the scalar product is associative, t_a is also an isometry.

t_a is called the **Miyamoto involution** (also **Majorana involution** in case of Majorana algebras) associated to the axis a.

In the case of the Conway-Norton-Griess algebra V_M, the Miyamoto involutions are precisely those induced by the elements of type $2A$ in M (i.e. the involutions in the M whose centraliser in M is $2^{\cdot}BM$).

Miyamoto involutions play a central rôle in the theory of Majorana representations and their dependence on axes is axiomatised in the definition of Majorana representations.

Given

- a group G, and a G-invariant set of involutions T generating G,
- a Majorana algebra V, and a set of axes τ generating V,

a **Majorana representation** of G on V with respect to T and τ is a pair (ϕ, ψ) such that

M1 $\phi\colon G \to Aut(V)$ is a faithful representation of G on V,

M2 τ is G-invariant,

M3 $\psi\colon T \to \tau$ is an isomorphism of G-sets,

M4 for every $t \in T$, t^{ϕ} is the Miyamoto involution associated to the axis t^{ψ},

M5 further conditions arising from the embeddings of the dihedral subalgebras in V_M

The standard example of a Majorana representation is when G is a subgroup of M generated by a set T of involutions of type $2A$.

In this case the restriction of the action of M on V_M induces a Majorana representation of G on an appropriate subalgebra of VM. The Majorana representations obtained this way are said to be **induced by an embedding of G in M**.

Goal: for a pair (G, T), where G is a finite group generated by a G-invariant subset T of involutions, determine all Majorana representations (ϕ, ψ) of G with respect to the generating set T.

3. State of the art

1. From Sakuma's theorem it follows that dihedral groups D_n admit Majorana representations if and only if $n \leq 6$. They are 9 in total and they all arise from an embedding in M.
2. In a series of papers, published between 2010 and 2014, Castillo-Ramirez, Decelle, Franchi, Ivanov, Pasechnik, Seress, Shpectorov, and M. studied Majorana representations of the groups S_4, A_5, A_6, A_7, A_{12}, $L_3(2)$, $L_2(11)$, and HN.

In particular, we have a complete description of Majorana representations of S_4, A_5, A_6, A_7.

group	n. of representations	dimensions	embedding in M
S_4	4	$13, 9$ $13, 9$	*yes* *no*, $C_2 \times S_4 \leq M$
A_5	4	$26, 20$ $46, 21$	*yes* *no*, $C_2 \times A_5 \leq M$
A_6	2	$76, 70$	*yes*, *no*
A_7	1	196	*yes*

Symmetric groups

Theorem (Norton 1985)

M has a subgroup isomorphic to S_n if and only if $n \leq 12$.

A subgroup of M isomorphic to S_{12} is generated by a set of involutions of type $2A$ containing all the elements corresponding to bitranspositions in S_{12}. Hence, for $n \leq 12$, S_n has a Majorana representation over a set containing its bitranspositions.

Theorem (Franchi, Ivanov, M. 2016)

The symmetric group S_n has a Majorana representation over a set containing its bitranspositions if and only if $n \leq 12$.

As I mentioned dihedral algebras cannot be spanned just by axes, but need also n-axes for $3 \leq n \leq 5$. So, in order to get a basis of the whole algebra V generated by the set T^{ψ} one needs to compute the subspace spanned by these n-axes (and hope that the products do not produce other vectors). In this context 3-axes (and 4-axes) are of particular interest because by an argument of Seress, quoted in Castillo Ramirez and Ivanov A_{12}'s paper, in the representations of A_{12}, the linear span of axes and 3-axes contains also the linear span of 5-axes.

In case of standard Majorana representations of the Symmetric groups, 3-axes are determined by cyclic subgroups generated by 3-cycles and by permutations of type $(3, 3)$.

Theorem (Franchi, Ivanov, M. 2018)

With the above conditions, let V_T and V_{T_3} be the subspaces of V generated by the axial vectors associated to the bitranspositions and the 3-cycles of S_n respectively. Then

1. *(Castillo-Ramirez, Ivanov) if $n = 12$, then $V_{T_3} \leq V_T$, and V_T decomposes into irreducible $\mathbb{R}[S_n]$-submodules as follows*

$$1 \oplus S^{(11,1)} \oplus S^{(10,2)} \oplus S^{(9,3)} \oplus S^{(8,4)} \oplus S^{(8,2,2)};$$

2. *if $n = 11$, then $V_{T_3} \cap V_T \cong S^{(9,2)}$, and $V_{T_3} + V_T$ decomposes into irreducible $\mathbb{R}[S_n]$-submodules as follows*

$$1 \oplus 1 \oplus 2S^{(10,1)} \oplus 2S^{(9,2)} \oplus 2S^{(8,3)} \oplus S^{(7,4)} \oplus S^{(8,2,1)} \oplus S^{(7,2,2)};$$

3. *if $n \in \{8, 9, 10\}$, $V_{T_3} \cap V_T = \{0\}$ and $V_{T_3} + V_T$ decomposes into irreducible $\mathbb{R}[S_n]$-submodules as follows*

$$1 \oplus 1 \oplus 2S^{(n-1,1)} \oplus 3S^{(n-2,2)} \oplus 2S^{(n-3,3)} \oplus S^{(n-4,4)} \oplus S^{(n-3,2,1)} \oplus S^{(n-4,2,2}$$

4. Combinatorial methods

It is convenient to extend the scalars from $\mathbb{R}$ to $\mathbb{C}$ and consider positive definite hermitian forms instead of scalar products.

The starting argument is the following easy linear algebra exercise: Suppose

$$f : W \to V$$

is a surjective linear map of $\mathbb{C}$-vector spaces and σ is a positive definite hermitian form on V. Then

1. σ and f define a hermitian form σ_f on W by setting

$$\sigma_f(w_1, w_2) = \sigma(w_1^f, w_2^f)$$

2. $\ker(f) = rad(\sigma_f)$.

In the case of a **Majorana representation** (ϕ, ψ) of a group G on a Majorana algebra V with respect to T and τ, the isomorphism of G-sets

$$\psi\colon T \to \tau$$

induces in a natural way an epimorphism of $\mathbb{C}[G]$-modules

$$\overline{\psi}\colon \mathbb{C}^T \to V_\tau,$$

(where $\mathbb{C}^T$ is the $\mathbb{C}[G]$ permutation module over T and V_τ is the linear span of τ) so that

$$V_\tau \cong \mathbb{C}^T / \ker(\overline{\psi}).$$

Now let σ_T be the hermitian form on $\mathbb{C}^T$ defined by

$$\sigma_T(t_1, t_2) := \sigma(t_1^{\psi}, t_2^{\psi}).$$

By the above argument,

$$\ker(\overline{\psi}) = rad(\sigma_T),$$

so, to determine the structure of the module V_τ, one is reduced to

1. determine the permutation module $\mathbb{C}^T$ and
2. its submodule $rad(\sigma_T)$,

The first issue is attacked by studying the permutation representations of the group G the second is achieved by finding a diagonal form of the Gram matrix Γ associated to σ_T with respect to the basis T.

Since G acts as a group of isometries on V, the scalar product is constant on the orbitals of G on τ.

Thus the relevant information about Γ is given by the Norton-Sakuma Theorem, once the **shape** of the representation is fixed, i.e. once we have assigned to each orbital of T an isomorphism class of a dihedral algebra generated by the two axes relative to one (any) pair of involutions in that orbital.

Similar arguments apply when we add to τ also the sets of $3-$, $4-$, and $5-$axes. Recall that a n-axis depends uniquely on the cyclic subgroup of order n of the dihedral group generated by the Miyamoto involutions associated to the generating axes of the corresponding dihedral algebra.

In this case we have to extend the set T to a set $\overline{T}$ that includes (some of) the sets U_n of these cyclic subgroups.
Of course, we need the scalar products between these axes. A lot of information on this side is already given by Ivanov's $A_6 - A_7$ paper (2011). Also the case of the 3-axes for representations induced by an embedding in M has been accomplished by Chien Sheng Lim in his PhD-thesis (2016).

Unfortunately, a direct computation of the Gram matrix is possible for small groups, but with larger ones (e.g. A_8 is already large) the dimensions involved make this computation very hard, if not impossible, even by machine.

At this stage, the theory of association schemes provides a sort of compressing data algorithm that reduces computations to a much more manageable situation.

The key step is to compute, for each orbit S of $\overline{T}$, the generalised first eigenmatrix associated to the action of G on S.

This matrix is defined in terms of the adjacency matrices: given an orbital $\mathcal{O}$ of G on S, the adjacency matrix associated to $\mathcal{O}$, is the square matrix whose rows and columns are indexed by the elements of S and, for $s_i, s_j \in S$, its (s_i, s_j)-entry is 1 if $(s_i, s_j) \in \mathcal{O}$ and 0 elsewhere.

Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the orbitals of G on S, $A_1, \ldots, A_r$ be the corresponding adjacency matrices, and, for $i \in \{1, \ldots, r\}$, let λ_i be the constant value of σ_S on $\mathcal{O}_i$.

The key features of the adjacency matrices are that,

1. $\Gamma = \sum_{i=1}^{r} \lambda_i A_i$.
2. these matrices form a basis for the **centraliser algebra** $C_{End(\mathbb{C}^S)}(G)$ of G over $\mathbb{C}^S$;

and, if the action of G on S is multiplicity-free,

3. the adjacency matrices are simultaneously diagonalisable by conjugation with a unitary matrix, say U, in particular $U^{-1}\Gamma U$ is a diagonal matrix whose set of diagonal entries coincides with the set of the eigenvalues of the adjacency matrices.
4. for each adjacency matrix A_j, the set of eigenvalues of A_j coincides with the set of the eigenvalues of the endomorphism $ad(A_j)$ of the centraliser algebra induced by right multiplication by A_j.

In case the action is not multiplicity-free one has to substitute "diagonalisable" in [3] with "block diagonalisable" where blocks correspond to the homogeneous components of G over $\mathbb{C}^S$ and, instead of eigenvalues in [4], take **generalised eigenvalues** which are not scalars, but endomorphisms of the corresponding homogeneous component.

So, in order to get the diagonal entries of $U^{-1}\Gamma U$, we can work inside the centraliser algebra and find a basis for the centraliser algebra that simultaneously block diagonalises the matrices $ad(A_j)$. The advantages are that

1. the dimension of the centraliser algebra is the number r of the orbitals of G on S, which is, in general, much smaller than the dimension $|S|$ of the permutation module $\mathbb{C}^S$.
2. the structure constants of the centraliser algebra are precisely the **intersection numbers** of the association scheme corresponding to the action of G on S which can be easily computed in terms of the orbitals.

More precisely:

$$A_i A_j = \sum_{k=1}^{r} p_{ij}^k A_k$$

where, for $(x, y) \in \mathcal{O}_k$,

$$p_{ij}^k = |\{z \in S | (x, z) \in \mathcal{O}_i \text{ and } (z, y) \in \mathcal{O}_j\}|.$$

in other words, the matrix associated to $ad(A_j)$ with respect to basis $A_1, \ldots, A_r$ of the centraliser algebra is the $r \times r$ matrix B_j, whose (i, k)-entry is precisely p_{ij}^k

The information about the eigenvalues of the adjacency matrices is concentrated in the **generalised first eigenmatrix**, i.e. the matrix P whose rows are indexed with the homogeneous components of the action of G on S, whose columns are indexed with the adjacency matrices of G on S, and whose (i, j)-entry is the generalised eigenvalue of the adjacency matrix corresponding to the column j on the homogeneous component corresponding to the row i.

In practice, one does not have to compute all the intersection matrices and simultaneously diagonalise them to get the generalised first eigenmatrix.

There are combinatorial methods that, in many cases, allow an easy computation of this matrix. In particular, the rows and columns of the generalised first eigenmatrix satisfy certain orthogonality relations (see Bannai-Ito for the multiplicity-free case, and, in the general case, Higman(1975) for the rows orthogonality relations, and Ivanov-Franchi-M (2016) for the columns orthogonality relations).

In a similar way character orthogonality relations can be used to complete the character table of a group, these relations enable to complete the generalised first eigenmatrix from partial information about its entries, at least if the action is multiplicity-free or close to it.

Once the generalised first eigenmatrix is achieved, one can easily get the diagonal form $U^{-1}\Gamma U$, and not just the dimension of $rad(\sigma_T)$ but also the decomposition of $rad(\sigma_T)$ into irreducible $\mathbb{C}[G]$-submodules.

Another key feature of the generalised first eigenmatrix is that, if the action of G on S is transitive, knowing the diagonal entries of its blocks one can compute the projection of an element of $\mathbb{C}^S$ into the irreducible submodules of $\mathbb{C}^S$.

More precisely if

$$V = V_1 \oplus \ldots \oplus V_s$$

is a decomposition of V into irreducible submodules and $(P_{hk})_{ii}$ is the i-th diagonal entry of the block P_{hk} of the first eigenmatrix corresponding to the orbital $\mathcal{O}_h$ and the irreducible submodule V_k, then the projection x_k of an element x of S into V_k with respect to the above decomposition is given by

$$x_k = \sum_{i=1}^{r} \frac{\dim(V_k)}{|\mathcal{O}_h|} (P_{hk})_{ii} \Big(\sum_{y \in \Delta_h(x)} y \Big),$$

where $\Delta_h(x) = \{y \in S | (x, y) \in \mathcal{O}_h\}$

This is particularly useful when the action of G is not transitive, e.g. when, in order to span the algebra V, together with τ, one needs to include also the n-axes.

Since an n-axis u is determined by a certain cyclic subgroup C_u of G, in this case one has to consider the action induced by G via conjugation on the set $T \cup T_n$ where T_n is the set of conjugates of C_u.

This action is clearly not transitive, whence not multiplicity-free (the trivial module e.g. has multiplicity at least 2), so the above procedure applied to the permutation module $\mathbb{C}^{T \cup T_n}$ does not necessarily work.

In this case we can split the permutation module into the direct sum of $\mathbb{C}^{T}$ and $\mathbb{C}^{T_n}$, restrict the above procedure to $\mathbb{C}^{T}$ and $\mathbb{C}^{T_n}$ and compare the irreducible modules appearing in each of these components.

Suppose e.g. that W_T and W_{T_n} are two copies of an irreducible $\mathbb{C}[G]$-module W appearing in $\mathbb{C}^T$ and $\mathbb{C}^{T_n}$ respectively. The above formula allows us to compute a set of generators of W_T and a set of generators of W_{T_n}.

Ideally one would like to compute the Gram matrix with respect of the union of those two sets of generators. However this can be awkward for these modules can have pretty large dimensions.

A better way is to find a subgroup H of G, such that

$$dim(C_{W_T}(H)) = dim(C_{W_{T_n}}(H)) = 1,$$

and check the Gram matrix with respect to a generator w_T of $C_{W_T}(H)$ and a generator w_{T_n} of $C_{W_{T_n}}(H)$.

Clearly, this 2×2 matrix has rank 1 if and only if $W_T = W_{T_n}$.