# Around Symmetries of Vertex Operator Algebras

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# §1 Vertex algebras and vertex operator algebras

— R.E. Borcherds: Vertex algebras, Kac-Moody algebras and the monster, Proc. Nat. Acad. Sci. U.S.A. 83, (1986), 3068–3071.

— I. Frenkel, J. Lepowsky, and A. Meurman: Vertex operator algebras and the Monster, Academic Press, New York, 1988.

**Vertex algebras.** A *vertex algebra* is a vector space $V$, say over $\mathbf{C}$, equipped with

- A bilinear map $V \times V \longrightarrow V((z))$, $(a,b) \mapsto \sum_{n\in\mathbf{Z}} a_{(n)} b z^{-n-1}$.
- A nonzero vector $\mathbf{1} \in V$, called the *vacuum vector*.

satisfying the following axioms:

(1) Borcherds identities. The following holds for all $p, q, r \in \mathbf{Z}$,

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)} b)_{(p+q-i)} c$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \Big( a_{(p+r-i)} (b_{(q+i)} c) - (-1)^r b_{(q+r-i)} (a_{(p+i)} c) \Big).$$

(2) Creation property.

$$a_{(n)} \mathbf{1} = \begin{cases} 0, & (n \geq 0), \\ a, & (n = -1). \end{cases}$$

**Vertex operators.** A vertex algebra is specified by the triple $(V, Y, \mathbf{1})$, where

$$Y : V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]], \quad a \mapsto Y(a, z) = \sum_n a_{(n)} z^{-n-1}.$$

The generating function $Y(a, z)$ is often called the vertex operator, although this term is also used in broader context.

A particular case of the axiom (1) takes the form

(1)′ For each $a, b \in V$, there exists a $d \in \mathbf{Z}_{\geq 0}$ such that

$$Y(a, z)Y(b, w)(z - w)^d = Y(b, w)Y(a, z)(z - w)^d.$$

It follows from the axioms (1)(2) that

(2)′
$$\mathbf{1}_{(n)} a = \begin{cases} 0, & (n \neq -1), \\ a, & (n = -1). \end{cases}$$

In other words, we have $Y(\mathbf{1}, z) = \mathrm{id}_V$.

**Translation operator.** Introduce the *translation* operator

$$T : V \longrightarrow V, \quad a \mapsto Ta = a_{(-2)}\mathbf{1}.$$

Then it follows from the axioms that:

(3) $a_{(-n-1)}\mathbf{1} = \dfrac{T^n a}{n!}$ for all $n \geq 0$.

(4) $T(a_{(n)}b) = (Ta)_{(n)}b + a_{(n)}(Tb)$ for all $n \in \mathbf{Z}$.

Proprety (4) says that $T$ is a derivation for all the products.

**Virasoro algebra.** The *Virasoro algebra* is the Lie algebra

$$\mathrm{Vir} = \bigoplus_{n \in \mathbf{Z}} \mathbf{C} L_n \oplus \mathbf{C} C$$

with the Lie bracket given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C, \quad [L_n, C] = 0.$$

When the central element $C$ of the Virasoro algebra acts by a scalar on a represetation $V$ of Vir, the scalar is called the *central charge* of $V$.

**Virasoro vector.** Let $V$ be a vertex algebra and $\omega \in V$ an element. Set $L_n = \omega_{(n+1)}$ for $n \in \mathbf{Z}$. Then $L_n$ satisfy the commutation relation of Virasoro algebra of central charge $c$ if and only if

$$\omega_{(n)}\omega = \begin{cases} 0 & (n \geq 4) \\ (c/2)\mathbf{1} & (n = 3) \\ 0 & (n = 2) \\ 2\omega & (n = 1) \\ T\omega & (n = 0) \end{cases}$$

Let us call such an $\omega$ a *Virasoro vector* of $V$ with central charge $c$.

**Vertex operator algebra.** A *vertex operator algebra* (VOA) is a graded vector space

$$V = \bigoplus_{\Delta \in \mathbf{Z}} V_\Delta, \quad \text{with } V_\Delta = 0 \text{ for } \Delta << 0$$

equipped with $Y : V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]]$, $\mathbf{1} \in V_0$ and $\omega \in V_2$ such that

(1) $(V, Y, \mathbf{1})$ is a vertex algebra. (2) $\omega$ is a Virasoro vector.

(3) $L_{-1} = T$. (4) $V_\Delta = \operatorname{Ker}(L_0 - \Delta)$ and $\dim V_\Delta < \infty$ for all $\Delta$

We then have the following property:

(5) $(V_{\Delta_1})_{(n)}(V_{\Delta_2}) \subset V_{\Delta_1+\Delta_2-n-1}$.

In particular, for $a \in V_n$, the operator $o(a) = a_{(n-1)}$ preserves the degree:

$$o(a) : V_\Delta \longrightarrow V_\Delta \quad \text{for all } \Delta.$$

**VOA of CFT type.** A VOA $V = \bigoplus_{\Delta} V_{\Delta}$ is said to be of <u>*CFT type*</u> if $V_{\Delta} = 0$ for $\Delta < 0$ and $V_0 = \mathbf{C}\mathbf{1}$. Set $\mathfrak{g} = V_1$:

$$\begin{aligned} V &= V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots . \\ &= \mathbf{C}\mathbf{1} \oplus \ \mathfrak{g} \ \oplus V_2 \oplus V_3 \oplus \cdots . \end{aligned}$$

Consider such a VOA and define operations $[-,-]$ and $(-\,|\,-)$ as follows:

$$[x, y] = x_{(0)}y, \quad (x\,|\,y)\mathbf{1} = x_{(1)}y, \quad \text{for} \quad x, y \in \mathfrak{g} = V_1.$$

Then $\mathfrak{g}$ is a Lie algebra with the Lie bracket $[-,-]$ and the form $(-\,|\,-)$ is a symmetric invariant bilinear form on $\mathfrak{g}$ in the sense that it satisfies

$$([x, y]\,|\,z) = (x\,|\,[y, z])$$

for all $x, y, z \in \mathfrak{g}$.

**Griess algebra of a VOA.** Consider a VOA of CFT type with $V_1 = 0$ and set $B = V_2$:

$$\begin{aligned} V &= V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots . \\ &= \mathbf{C1} \oplus \ 0 \ \oplus B \ \oplus V_3 \oplus \cdots . \end{aligned}$$

Define operations $\cdot$ and $(-|-)$ on $B = V_2$ by

$$x \cdot y = x_{(1)} y, \quad (x|y)\mathbf{1} = x_{(3)} y, \quad \text{for} \quad x, y \in B = V_2.$$

Then $B$ becomes a commutative nonassociative algebra with respect to the product $\cdot$, and $(-|-)$ is a symmetric invariant bilinear form in the sense that, for all $x, y, z \in B$,

$$(x \cdot y|z) = (x|y \cdot z).$$

The triple $\big(B, \cdot, (-|-)\big)$ is called the <u>*Griess algebra*</u> of $V$.

**Moonshine module.** The famous moonshine module $V^\natural$ is a VOA considered by Borcherds and studied in detail by Frenkel–Lepowsky–Meurman (FLM). It is of CFT type with $V_1^\natural = 0$. Its shape is as

$$V^\natural = \underline{1} \oplus 0 \oplus \underline{196884} \oplus \underline{21493760} \oplus \underline{864299970} \oplus \cdots ,$$

whose dimensions are in agreement with the coefficients of the $J$-invariant

$$J(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots .$$

FLM gave a constructon of $V^\natural$ and proved that the automorphism group is the Monster, the largest sporadic finite simple group.

Indeed, the algebra $B^\natural = V_2^\natural$ agrees with the algebra considered by Conway in 1985, a variant of the algebra considered earlier by Griess in constructing the Monster. We call it the *Griess–Conway algebra*.

The VOA $V^\natural$ gave rise to a natural reformulation of the moonshine conjecture of Conway–Norton, which is solved by Borcherds in 1992.

## Further references

— V. Kac: Vertex algebras for beginners. University Lecture Series 10, Amer. Math. Soc., Providence, RI, 1998.

— A. Matsuo and K. Nagatomo: Axioms for a vertex algebra and the locality of quantum fields, MSJ Memoirs 4, Mathematical Society of Japan, Tokyo, 1999.

— J. Lepowsky and H.S. Li: Introduction to Vertex Operator Algebras and Their Representations. Progress in Mathematics 227, Birkhäuser, 2004.

# §2 Norton's trace formulae and conformal designs.


— S.P. Norton: The Monster algebra: Some new formulae. In: Moonshine, the Monster, and related topics (South Hadley, MA, 1994), Contemp. Math. 193, Amer. Math. Soc., 1996, 297–306.

— A. Matsuo: Norton's trace formulae for the Griess algebra of a vertex operator algebra with larger symmetry. Commun. Math. Phys. 224 (2001), 565–591.

— M. Tuite: Exceptional vertex operator algebras and the Virasoro algebra. Vertex operator algebras and related areas, 213–225, Contemp. Math., 497, Amer. Math. Soc., Providence, RI, 2009.

— G. Höhn: Conformal designs based on vertex operator algebras. Advances in Math. 217, (2008), 2301–2335.

**Theorem (S. Norton, 1996)** On the Griess–Conway algebra $B^\natural$, let $\mathrm{ad}(a) : B^\natural \longrightarrow B^\natural$ denote the multiplication by $a$. Then, in our notations and conventions, the following formulae hold:

$$\begin{aligned}
\mathrm{Tr}\,\mathrm{ad}(a) &= 32814(a\,|\,\omega),\\
\mathrm{Tr}\,\mathrm{ad}(a_1)\mathrm{ad}(a_2) &= 4620(a_1\,|\,a_2) + 5084(a_1\,|\,\omega)(a_2\,|\,\omega),\\
\mathrm{Tr}\,\mathrm{ad}(a_1)\mathrm{ad}(a_2)\mathrm{ad}(a_3) &= \cdots\,,\\
\mathrm{Tr}\,\mathrm{ad}(a_1)\mathrm{ad}(a_2)\mathrm{ad}(a_3)\mathrm{ad}(a_4) &= \cdots\,,\\
\mathrm{Tr}\,\mathrm{ad}(a_1)\mathrm{ad}(a_2)\mathrm{ad}(a_3)\mathrm{ad}(a_4)\mathrm{ad}(a_5) &= \cdots\,,
\end{aligned}$$

with an additional totally anti-symmetric quintic form for the last one.

**Vertex algebra with large symmetry.** Let $V$ be a VOA and consider the subVOA generated by $\omega$:

$$V_\omega = \mathbf{C}\mathbf{1} \oplus 0 \oplus \mathbf{C}\omega \oplus \mathbf{C}T\omega \oplus \cdots .$$

Then the elements of $V_\omega$ are fixed by the action of $\operatorname{Aut} V$:

$$V_\omega \subset V^{\operatorname{Aut} V}.$$

Let us say that the VOA $V$ is <u>*of class* $S^n$</u> if the equality holds on the homogeneous subspaces up to degree $n$:

$$V_\omega \cap V_\Delta = V^{\operatorname{Aut} V} \cap V_\Delta \quad \text{for} \quad \Delta \leq n.$$

The moonshine module $V^\natural$ is of class $S^{11}$.

**Theorem (A.M., 2001)** Let $V$ be a VOA of CFT type with $V_1 = 0$ and $d = \dim B = \dim V_2$. Then, under some reasonable assumptions:

(1) If $V$ is of class $S^2$, then $\mathrm{Tr}\,\mathrm{ad}(a) = \dfrac{4d}{c}(a\,|\,\omega)$.

(2) If $V$ is of class $S^4$, then $\mathrm{Tr}\,\mathrm{ad}(a_1)\mathrm{ad}(a_2)$ is expressed as

$$\frac{-2(5c^2 - 88d + 2cd)}{c(5c+22)}(a_1\,|\,a_2) + \frac{4(5c+22d)}{c(5c+22)}(a_1\,|\,\omega)(a_2\,|\,\omega).$$

If $V$ is of class $S^6$, $S^8$ and $S^{10}$, then the traces $\mathrm{Tr}\,\mathrm{ad}(a_1)\mathrm{ad}(a_2)\mathrm{ad}(a_3)$, $\mathrm{Tr}\,\mathrm{ad}(a_1)\mathrm{ad}(a_2)\mathrm{ad}(a_3)\mathrm{ad}(a_4)$ and $\mathrm{Tr}\,\mathrm{ad}(a_1)\mathrm{ad}(a_2)\mathrm{ad}(a_3)\mathrm{ad}(a_4)\mathrm{ad}(a_5)$ are expressed in similar ways, respectively, with an additional totally anti-symmetric quintic form for the last one.

For $V = V^\natural$, the formulae above agrees with Norton's formulae. The formula for a general $V$ is called *Matsuo–Norton trace formula* by some authors.

**Consequences of the symmetry condition.** Let $V$ be as above. Under some reasonable assumptions, the following hold:

(1) Let $x_1, \ldots, x_d$ be a basis of $B$ and $x^1, \ldots, x^d$ be the dual basis with respect to the bilinar form. Consider the 'Casimir element'

$$\kappa_d = \sum (x^i)_{(3-d)}(x_i).$$

If $V$ is of class $S^n$ then $\kappa_d \in V_\omega$ for all $d \leq n$.

(2) Consider the projection map $\pi : V \longrightarrow V_\omega$. If $V$ is of class $S^n$ then

$$\mathrm{Tr}\,|_{V_\Delta} o(a) = \mathrm{Tr}\,|_{V_\Delta} o(\pi(a))$$

holds for all $a \in V_d$ with $d \leq n$.

M. Tuite took the former property as the definition of 'exceptional VOAs' and G. Höhn the latter as that of 'conformal design.'

**Conformal designs (Höhn, 2008)** A homogeneous subspace $V_\Delta$ of a VOA $V$ is said to be a *conformal t-design* if

$$\mathrm{Tr}\,|_{V_\Delta} o(a) = \mathrm{Tr}\,|_{V_\Delta} o(\pi(a))$$

holds for all $a \in V_n$ with $n \leq t$.

According to Höhn, conformal designs are analogous to spherical designs (and also to block designs), because a subset $X$ of the unit sphere in $\mathbf{R}^n$ is a spherical $t$-design if and only if

$$\sum_{B \in X} f(B) = \sum_{B \in X} \pi(f)(B).$$

holds for all homogeneous polynomial $f$ of degree $\leq t$, where $\pi$ is the projection of the space of homogeneous polynomials onto the trivial component with respect to the action of $\mathrm{O}_n(\mathbf{R})$. Höhn actually defines conformal designs more generally for a homogeneous subspace of $V$-module.

## Further references

— H. Yamauchi: Extended Griess algebras and Matsuo–Norton trace formulae. In Conformal field theory, automorphic forms and related topics (Heidelberg, Germany, 2011), ed. W. Kohnen and R. Weissauer, Springer, Berlin, 2014, 75–107.

— T. Hashikawa: Conformal designs and minimal conformal weight spaces of vertex operator superalgebras. Pacific J. Math. Vol. 284, (2016), No. 1, 121–145.

# §3 Algebras associated with partial triple systems.

— J.H. Conway: A simple construction for the Fischer-Griess monster group. Invent. Math. 79, (1985), 513–540.

— M. Miyamoto: Griess algebras and conformal vectors in vertex operator algebras. J. Algebra 179, Issue 2, (1996), 523–548.

— A. Matsuo: 3-transposition groups of symplectic type and vertex operator algebras. Preprint version. arXiv:math/0311400.

— A. Matsuo: 3-transposition groups of symplectic type and vertex operator algebras. Published version. J. Math. Soc. Japan 57 (3), (2005), 639–649.

**Idempotents in Griess algebras.** Let $V$ be a VOA of CFT type with $V_1 = 0$ and set $B = V_2$:

$$V = \mathbf{C1} \oplus 0 \oplus B \oplus V_3 \oplus \cdots$$

If $e \in B$, then

$$e \text{ is a Virasoro vector} \iff e \cdot e = 2e$$

Thus Virasoro vectors and idempotents are in one-to-one correspondence in $B$ by $e \leftrightarrow e/2$. We denote the corresponding action by

$$L^e_n = e_{(n+1)}, \quad n \in \mathbf{Z},$$

whose central charge is given by

$$c_e = 2(e|e).$$

Let $\mathrm{Vir}(e)$ denote the action of the Virasoro algebra via the operators $L^e_n$, $n \in \mathbf{Z}$.

**Ising vectors.** A Virasoro vector $e$ is called an *Ising vector* if the Virasoro representation generated by $\mathbf{1}$ with respect to the action $\mathrm{Vir}(e)$ is isomorphic to the irreducible representation with central charge $1/2$ and lowest conformal weight 0. Namely,

$$U(\mathrm{Vir}(e))\mathbf{1} \cong L(1/2, 0),$$

Thus

$$e \in B \text{ is an Ising vector } \Longrightarrow e \cdot e = 2e \text{ and } (e|e) = 1/4.$$

Under existence of positive-definite invariant Hermitian form, the converse also holds for real Virasoro vectors. From now on, we assume that such conditions hold.

**Consequences of Virasoro representation theory.** Let $e \in B$ be an Ising vector. Then, by the representation theory of Virasoro algebra, we may decompose $V$ with respect to the action $\mathrm{Vir}(e)$ as

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16),$$

where $V_e(h)$ is the sum of irreducible representations with lowest conformal weight $h$.

**Theorem (Miyamoto)** The following map is an automorphism of $V$:

$$\tau_e : V \longrightarrow V, \quad x \mapsto \tau_e(x) = \begin{cases} 1, & (x \in V_e(0) \oplus V_e(1/2)), \\ -1, & (x \in V_e(1/16)). \end{cases}$$

If $V_e(1/16) = 0$, then the following map is an automorphism of $V$:

$$\sigma_e : V \longrightarrow V, \quad x \mapsto \sigma_e(x) = \begin{cases} 1, & (x \in V_e(0)), \\ -1, & (x \in V_e(1/2)). \end{cases}$$

These automorphisms are often called *Miyamoto involutions*.

**Theorem (Miyamoto)** Let $S$ be a set of Ising vectors in $B$ and assume $V_e(1/16) = 0$ for every $e \in S$. Set $D = \{ \sigma_e \mid e \in S \}$ and $G = \langle D \rangle$. Then $(G, D)$ is a 3-transposition group.

Here a 3-transposition group is a pair $(G, D)$ of a group $G$ and a normal subset $D$ of involutions which generates $G$ such that, for every pair $g, h \in G$, $|gh| \leq 3$.

For instance, the symmetric group $G = \mathrm{Sym}_n$ becomes a 3-tranposition group by taking $D$ to be the set of transpositions $(ij)$.

**Properties of Ising vectors (Miyamoto)** Let $e, f \in B$ be Ising vectors such that $V_e(1/16) = V_f(1/16) = 0$. Then one of the following holds:

(1) $e = f$ (hence $e \cdot f = 2f$ and $(e|f) = 1/4$).

(2) $e \cdot f = 0$ and $(e|f) = 0$.

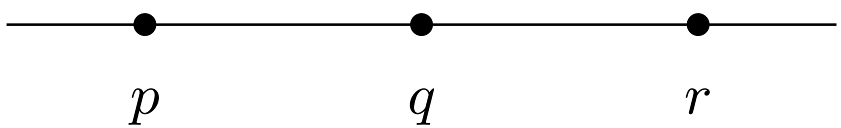
(3) $e \cdot f = (e + f - g)/4$ and $(e|f) = 1/32$.

In (3), $g = \sigma_e(f) = \sigma_f(e)$, which is again an Ising vector.

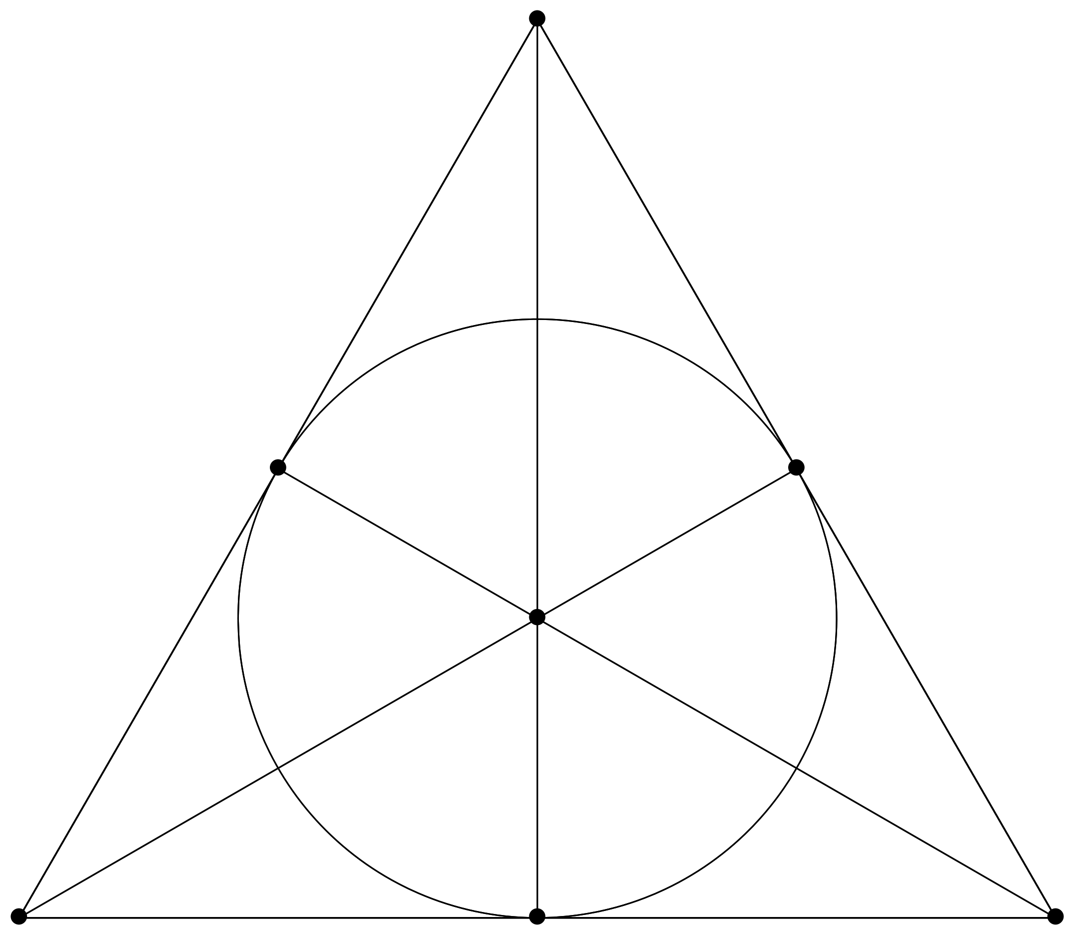
**Partial triple system.** Let $\Omega$ be a set and let $\mathcal{L}$ be a set of subsets of $\Omega$. Call an element of $\Omega$ a point and an element of $\mathcal{L}$ a line. The pair $(\Omega, \mathcal{L})$ is a *partial triple system* (or a *partial linear space of order* 2) if the following conditions are satisfies:

(1) Any line consists of exactly three points.

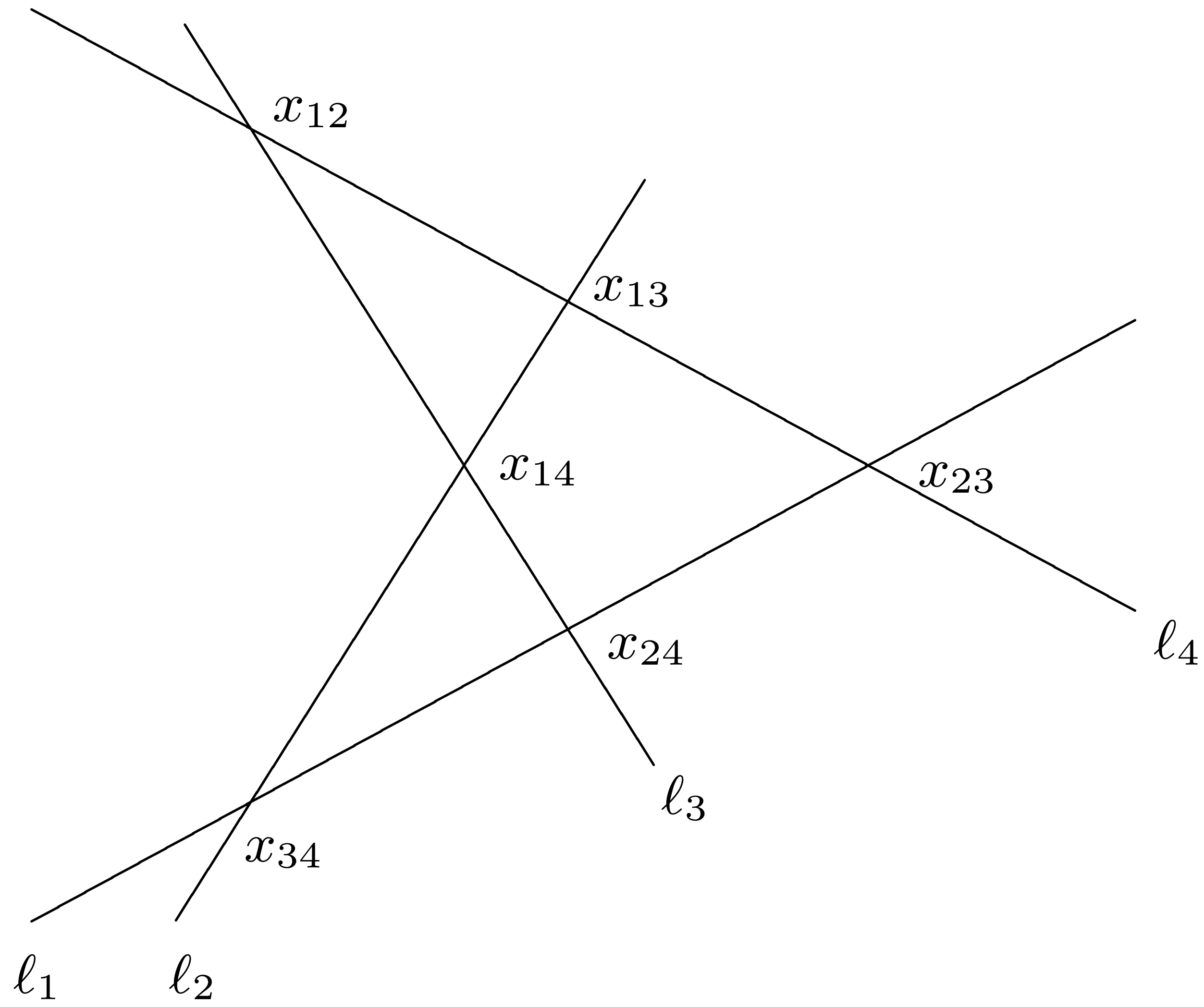
(2) Any two distinct points belong to at most one line.



# Fano plane.



# Dual affine plane of order 2.



**Colinearity graph.** Distinct points are said to be *colinear* if they belong to the same line. Pairs $(p, q)$ of points of a partial triple system fall into the following cases:

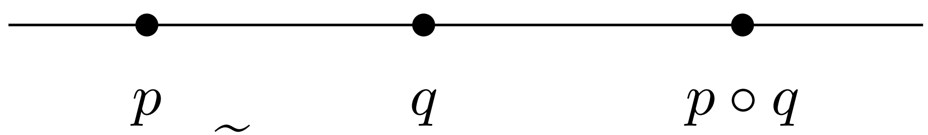
(1) $p = q$ ‘$p = q$’

(2) $p \neq q$ and $p, q$ are not colinear. ‘$p \perp q$’

(3) $p \neq q$ and $p, q$ are colinear. ‘$p \sim q$’

Consider the graph with the vertices being the points and two points $p, q$ are connected by an edge when they are colinear. This graph is called the *colinearity graph* of the partial triple system.

**Associated algebra.** Let $(\Omega, \mathcal{L})$ be a partial triple system. For $p \sim q$, let $p \circ q$ denote the other point of the line passing through $p$ and $q$.




Consider the vector space $\widetilde{B}$ spanned by the symbols $e_p$, $p \in \Omega$.

$$\widetilde{B} = \bigoplus_{p \in \Omega} \mathbf{C} e_p.$$

Define the multiplication and scalar product by the following table:

| — | $e_p \cdot e_q$ | $(e_p \vert e_q)$ |
|---|---|---|
| $p = q$ | $2e_p$ | $\gamma/2$ |
| $p \perp q$ | $0$ | $0$ |
| $p \sim q$ | $\delta(e_p + e_q - e_{p \circ q})/2$ | $\delta\gamma/8$ |

In this way, we obtain a commutative nonassociative with a symmmetric invariant bilinear form (called the *Matsuo algebra* by some authors).

**Fischer space of symplectic type.** A *Fischer space* is a partial triple system such that any plane is isomorphic to the dual affine plane of order 2 or the affine plane $\mathbf{A}^2(3) = \mathbf{F}_3^2$ of order 3. There is a one-to-one correspondence between Fischer spaces and 3-tranpsosition groups in a certain sense.

A Fischer space is said to be of *symplectic type* if any plane is isomorphic to the dual affine plane of order 2.

**Theorem (A.M., 2005)** The 3-transposition groups that arise as Miyamoto described are of symplectic type with the least eigenvalue of the colinearity graph $\geq -8$.

| $G$ | $\nu$ | $k$ | $s$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $\mathrm{Sym}_3$ | $3$ | $2$ | $-1$ | $6/5$ | $3$ |
| $\mathrm{Sym}_n$ | $n(n-1)/2$ | $2n-4$ | $-2$ | $n(n-1)/(n+2)$ | $n(n-1)/2$ |
| $E:\mathrm{Sym}_n$ | $n(n-1)$ | $4n-8$ | $-4$ | $n-1$ | $n(n-1)$ |
| $E^2:\mathrm{Sym}_n$ | $2n(n-1)$ | $8n-16$ | $-8$ | $n$ | $(3n-1)n/2$ |
| $\mathrm{O}_6^-(2)$ | $36$ | $20$ | $-4$ | $36/7$ | $36$ |
| $\mathrm{Sp}_6(2)$ | $63$ | $32$ | $-4$ | $63/10$ | $63$ |
| $\mathrm{O}_8^+(2)$ | $120$ | $56$ | $-4$ | $15/2$ | $120$ |
| $2^6:\mathrm{O}_6^-(2)$ | $72$ | $40$ | $-8$ | $6$ | $57$ |
| $2^6:\mathrm{Sp}_6(2)$ | $126$ | $64$ | $-8$ | $7$ | $91$ |
| $2^8:\mathrm{O}_8^+(2)$ | $240$ | $112$ | $-8$ | $8$ | $156$ |
| $\mathrm{O}_8^-(2)$ | $136$ | $72$ | $-8$ | $34/5$ | $85$ |
| $\mathrm{Sp}_8(2)$ | $255$ | $128$ | $-8$ | $15/2$ | $120$ |
| $\mathrm{O}_{10}^+(2)$ | $496$ | $240$ | $-8$ | $8$ | $156$ |

## Further references

— A.A. Ivanov, The Monster Group and Majorana Involutions, Cambridge Tracts in Mathematics, vol. 176, 2009.

— J.I. Hall, F. Rehren, and S. Shpectorov: Primitive axial algebras of Jordan type. Journal of Algebra 437, (2015) 79–115.

# §4 Acknowledgement

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