Symmetries and Combinatorics of Finite Antilattices

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Well, I have a suggestion:

Is this Siberia? I discovered this only yesterday!

Lattice

A *lattice* is a set N equipped with operations $\wedge$, $\vee$, called meet and join, that satisfy the following laws (associativity, idempotence):

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \tag{1}$$

$$a \wedge a = a \tag{2}$$

$$a \vee (b \vee c) = (a \vee b) \vee c \tag{3}$$

$$a \vee a = a \tag{4}$$

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$$a \vee (b \vee c) = (a \vee b) \vee c \tag{3}$$

$$a \vee a = a \tag{4}$$

commutativity and two dual absorption laws:

$$a \wedge b = b \wedge a \tag{5}$$

$$a \vee b = b \vee a \tag{6}$$

$$a \vee (a \wedge b) = a \tag{7}$$

$$a \wedge (a \vee b) = a \tag{8}$$

Non-commutative Lattice, Antilattice and Quasilattice

An algebra N equipped with two operations $\wedge$ and $\vee$ satisfying (1–4) from the previous slide is called *non-commutative lattice* or *double band*.

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A non-commutative lattice is called an *antilattice* if its operations $\wedge$, $\vee$ satisfy also the following dual absorption laws:

$$a \wedge b \wedge a = a. \tag{9}$$

$$b \vee a \vee b = b. \tag{10}$$

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A non-commutative lattice is called an *antilattice* if its operations $\wedge$, $\vee$ satisfy also the following dual absorption laws:

$$a \wedge b \wedge a = a. \tag{9}$$

$$b \vee a \vee b = b. \tag{10}$$

By replacing (9–10) with

$$b \vee a \vee b = b \iff a \wedge b \wedge a = a. \tag{11}$$

we get the axioms for a *quasilattice*.

Magmas, Semigroups, Bands and Rectangular Bands

Set B equipped with a binary operation $\bullet$ is called *magma* $(B, \bullet)$.

Magmas, Semigroups, Bands and Rectangular Bands

Set B equipped with a binary operation $\bullet$ is called *magma* $(B, \bullet)$. Consider the following identities: associativity, idempotence and rectangularity.

$$a \bullet (b \bullet c) = (a \bullet b) \bullet c \tag{12}$$

$$a \bullet a = a \tag{13}$$

$$b \bullet a \bullet b = b. \tag{14}$$

- If $(B, \bullet)$ satisfies (12) it is called *semigroup*.
- If $(B, \bullet)$ satisfies (12–13) it is called *band*.
- If $(B, \bullet)$ satisfies (12–14) it is called *rectangular band*.

Excursion to Universal Algebra: Algebra and its signature.

Structures, such as bands, antilattices, quasilattices, semigroups, etc. may be considered as *algebras* in the sense of universal algebra. An important algebra invariant is its *signature*.

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Algebra	Signature	Examples
$(B, \bullet)$	(2)	magma, band, semigroup
$(M, \bullet, \epsilon)$	$(2, 0)$	monoid
$(N, \wedge, \vee)$	$(2, 2)$	antilattice, (non-commutative) lattice
$(N, \wedge, \vee, \neg, 0, 1)$	$(2, 2, 1, 0, 0)$	Boolean algebra

Variety of algebras.

As usual in universal algebra a *variety* of algebras is any collection of algebras of the same signature that is closed under:

- taking homomorphic images,
- taking substructures,
- taking (general) direct products.[3]

---

[3] If we replace general direct product by finite direct products we get a *pseudo-variety*. For instance finite algebras from a variety, form pseudo-variety.

Two fundamental theorems.

We need two very important theorems from the theory of universal algebras.

Birkhoff's HSP Theorem.

The first one is due to Garrett Birkhoff (1935) and gives a beautiful characterisation of varieties.

Theorem (Birkhoff's HSP Theorem)

A collection of algebras is a variety (i.e. it is closed under homomorphisms, subalgebras and products) if and only if it can be defined by a set of identities (equalities).

Importance of Birkhoff's HSP Theorem.

Since bands, rectangular bands, lattices, antilattices, non-commutative lattices, etc. can be defined by collections of identities, each of them forms a variety and there is no need to test closure under homomorphisms, substructures and products.

Quasilattices From a Variety

Recall that we defined quasilattices by a set of identities (1–4) and an axiom (11) that involves equivalence.

Proposition

In a non-commuttative lattice the axiom (11) is equivalent to the following pair of identities.

$$a \wedge (b \vee a \vee b) \wedge a = a \tag{15}$$

$$a \vee (b \wedge a \wedge b) \vee a = a \tag{16}$$

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Corollary

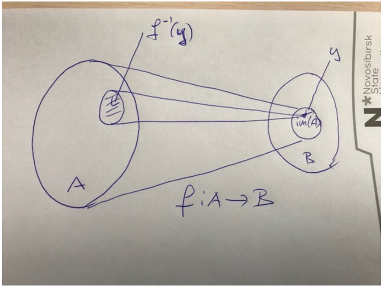
Quasilattices form a variety.

The First Isomorphism Theorem.

Theorem (First Isomorphism Theorem)

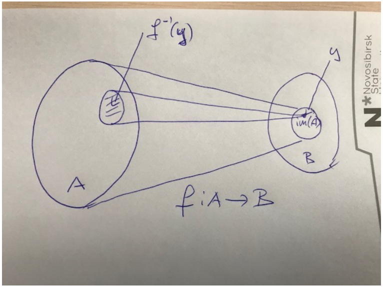
Let $f : A \to B$ be a mapping between algebras A and B of the same signature, with $im(f) \subseteq B$ its image and the set partition of A its kernel $ker(f)$, defining an equivalence relation $\Phi = ker(f)$ on A. Then f is a homomorphism if and only if Φ is a congruence on A, in which case the factor structure A/Φ is an algebra isomorphic to $im(f)$.

First Isomorphism Theorem in pictures



Homomorphism $f : A \to B$. Note that $y \in im(f) = im(A) \subseteq B$ and $f^{-1}(y) \subseteq A$.

First Isomorphism Theorem in pictures



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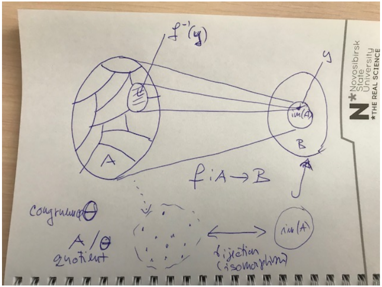

$ker(f) = \Theta = \{f^{-1}(y) | y \in \mathrm{im}(f)\}$ is a congruence on A. A/Θ is isomorphic to $im(f) = im(A)$.

Importance of the First Isomorphism Theorem.

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Importance of the First Isomorphism Theorem.

- Studying homomorphic images is the same thing as studying quotient structures
- and this is the same thing as studying congruences.
- In other words all information about potential homomorphic images of $f : A \to B$ resides in A alone.
- The role of B is to select from all potential homomorphcic images determined by A those that can be realized using B.

Three important lattices.

For a given algebra A we denote by

- $Equ(A)$ all equivalence relations on A.
- By $Con(A)$ we denote the set of all congruence relations and
- by $Sub(A)$ all substructures of A.

Each of them forms a lattice, actually, an algebraic[4] lattice.

[4] This fact is not relevant for our talk.

Congruences in bands.

Proposition

Let A be a band and θ any of its congruences. Thren each congruence class is a band,

Proof.

Let $[a]$ be a congruence class of θ and let $a, b \in [a]$. From $a\theta b$ it follows $aa\theta ab$ and $a\theta ab$. Hence $ab \in [a]$. $\square$

Note the **key role** in the proof played by the **idempotence**.

The subvarieties of bands. Regular bands.

Recall that a band is defined by identities (1–4). Let $\mathcal{V}$ denote the variety of bands. There are several important facts about the collection of all subvarieties of bands.

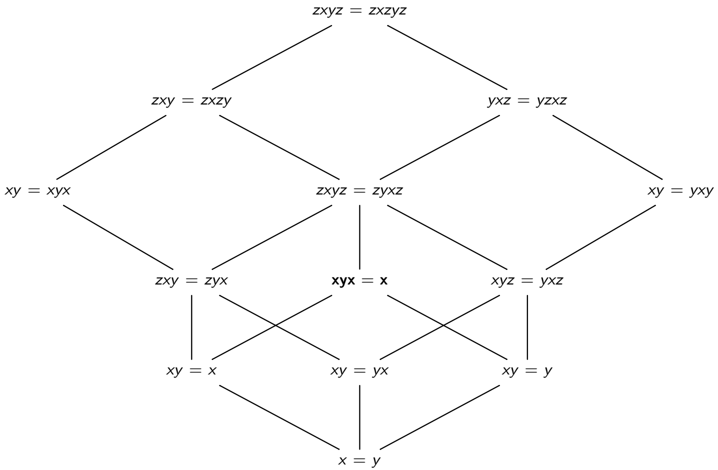
- Each subvariety $\mathcal{U}$ of bands is determined by a single additional identity.
- All subvarieties of bands form a lattice.
- Let $\mathcal{U}$ be any subvariety of bands. All subvarieties of $\mathcal{U}$ form a finite lattice.

Definition

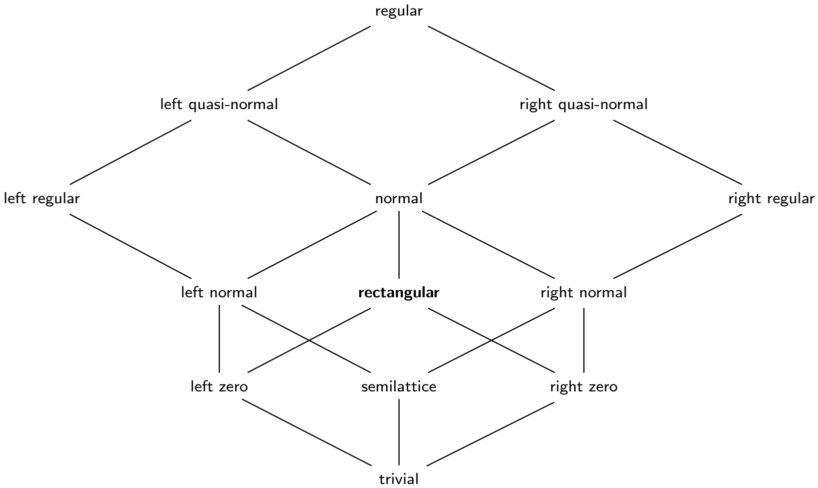
The following identity is called regularity and the band satisfying regularity is called *regular band*.

$$abca = abaca \tag{17}$$

The 13 subvarieties of regular bands



The 13 subvarieties of regular bands and the position of rectangular bands in it.



Generating matrix and type of rectangular bands

The operation $\bullet$ of a magma B on n elements is usually given by an $n \times n$ *Cayley table*. The best known algorithm for testing whether B is a band takes $O(n^3)$ steps: $O(n)$ for idempotence and $O(n^3)$ for associativity. For rectangular bands the test is much simpler due to the following result.

Proposition

A band B of order n is rectangular if and only if there exist p, q such that $n = pq$, and a $p \times q$ matrix G with distinct entries from B, called a generating matrix for B and $(G)_{ij} \bullet (G)_{kl} = (G)_{il}$.

Obviously, testing whether a band is rectangular can be done in $O(n^2)$ steps.
The pair (p, q) is an invariant of a rectangular band and will be called its *type*.

Isomorphism classes of rectangular bands.

Clearly, rectangular bands of different types are non-isomorphic. On the other hand, any two rectangular bands or the same type are isomorphic. Let *beta*(n) denote the number of isomorphism classes of rectangular bands of order n.

Proposition

There are $\beta(n) = \sigma_0(n)$ isomorphism classes of rectangular bands of order n, where $\sigma_0(n)$ is the number of divisors of n.

We will continue this story some other time.

Equivalent generating matrices.

Definition

Two generating matrices are *equivalent* if one can be obtained from the other one by simultaneously permuting rows and columns.

Proposition

Two generating matrices define the same Cayley table, and hence the same rectangular band if and only if they are equivalent.

Proposition

Any equivalence class of generating matrices of type (p, q) *contains* $p!q!$ *matrices.*

Corollary

There are $(pq)!/(p!q!)$ *distinct rectangular bands of type* (p, q)*.*

Standard form.

A generating matrix is in the *standard form* if

- It has the smallest element in the first position of the first row.
- The elements in the first row are in the increasing order.
- The elements in the first column is the increasing order.

Proposition

Any equivalence class of generating matrix has exactly one matrix in standard from. Two matrices in the standard form are equivalent if and only if they are equal.

Putting a matrix into the standard from.

By permuting rows and columns, one can place any generating matrix into the standard form. Both permutations are determined by the row and column of the minimal element in the matrix.

Original matrix:

$$\begin{array}{rrrrr} 17 & 16 & 18 & 19 & 20 \\ 5 & 4 & 3 & 2 & 1 \\ 11 & 7 & 8 & 9 & 10 \\ 6 & 12 & 13 & 14 & 15 \end{array}$$

Its standard form:

$$\begin{array}{rrrrr} 1 & 2 & 3 & 4 & 5 \\ 10 & 9 & 8 & 7 & 11 \\ 15 & 14 & 13 & 12 & 6 \\ 20 & 19 & 18 & 16 & 17 \end{array}$$

(Row) canonical form.

A generating matrix is in the *(row) canonical form* if the elements are sorted row-wise in the increasing order. In other words, the elements in each row are placed in increasing order, as are the elements in each column as you descend.
For instance, this forces:

1	2	3	4	5
6	7	8	9	10
		etc		

Proposition

Canonical form is standard. For each type (p, q) there is only one generating matrix of that type in the canonical form.

The automorphism group of a rectangular band.

Proposition

Let B be a finite rectangular band of type (p, q) given by one of its generating matrices G. Its group of automorphisms is isomorphic to $S_p \times S_q$ where the element $(\pi, \chi) \in S_p \times S_q$ defines the automorphism $\phi : B \to B$ with

$$\phi : G_{ij} \mapsto G_{\pi(i)\chi(j)}$$

Antilattice as a double rectangular band

A pair of bands defined on the same underlying set is called a *double band* (or *non-commutative lattice*). Any lattice or quasilattice is therefore a double band for $\wedge$ and $\vee$.
The converse is not true. It is not hard to find a double band that is neither a quasilattice nor a lattice.
For antilattices the situation is much simpler.

Proposition

Any rectangular double band is antilattice.

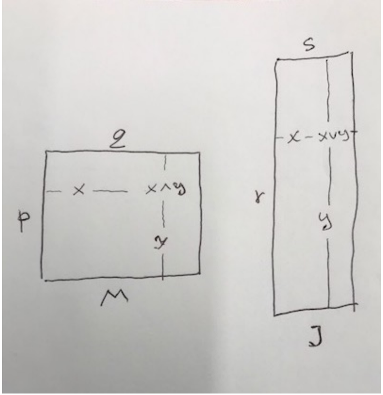
Proposition

If one of the two bands of a quasilattice N is rectangular, the other one is rectangular, too, and N is an antilattice.

Type of an antilattice N.

Since any antilattice N is given by a pair of generating matrices, say M (meet) of order (p, q), and J (join) of order (r, s), $pq = rs = n$, the quadruple $(p, q, r, s) = ((p, q), (r, s))$ is an antilattice invariant that will be called the *type* of N.

Antilattices in pictures



Each antilattice $N(\wedge, \vee)$ is determined by two generating matrices M and J. For any two elements $x, y \in N$ computation of $x \wedge y$ and $x \vee y$ is indicated. Note that $pq = rs$ is the order n of N.

Quasilattice as a lattice of antilattices

Theorem (Laslo–Leech analog of Clifford-McLean Theorem)

Any quasilattice is a lattice of antilattices.

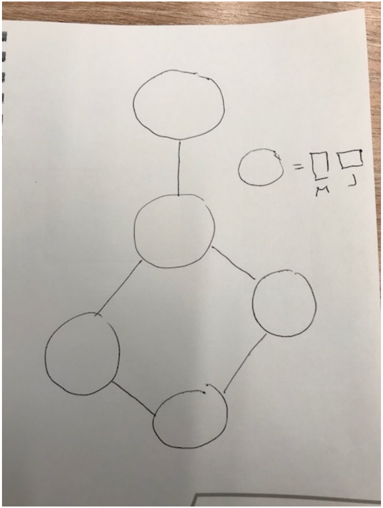
Proposition

A rectangular band is commutative if and only if it is of order 1.

Proposition

If one of the two bands of a quasilattice N is commutative, the other one is commutative, too, and N is a lattice.

Laslo-Leech Theorem in pictures



Each quasilattice $N(\wedge, \vee)$ is a lattice of antilattices.

Flat Antilattices.

If both generating matrices M and J are vectors, the antilattice N is called *flat*. There are four types of flat anitlattcies of order n:

type (p, q, r, s)	type name
$(1, n, 1, n)$	LL-flat
$(n, 1, 1, n)$	RL-flat
$(1, n, n, 1)$	LR-flat
$(n, 1, n, 1)$	RR-flat

Skew and skew* antilattices.

Definition

Let N be an antilattice. Then N is a *skew* antilattice if it can be generated by matrices M and J such that $J = M^T$.

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Let N be an antilattice. Then N is a *skew** antilattice if it can be generated by matrices M and J such that $J = M$.

Skew and skew* antilattices have essentially the same properties as their reducts.

Proposition

Let N be an antilattice of type (p, q, r, s). If N is skew then $(p, q, r, s) = (p, q, q, p)$. If N is skew then $(p, q, r, s) = (p, q, p, q)$. In each case $\mathrm{Aut}N$ is isomorphic to $S_p \times S_q$.*

Direct products.

Let N and N' be antilattices, of respective types (p, q, r, s) and (p', q', r', s'), determined by respecitve generating matrices pairs (M, J) and $M', J')$. The direct product $N \times N'$ is of type (pp', qq', rr', ss'), determined by the two tensor products of matrices $(M \otimes M', J \otimes J')$.
One has to clarify how the entries of these tensor products are computed. Namely, they are simply pairs of the form $(x, x') \in N \times N'$.

Example.

Here is an example.
Let N be determined by

$$M = \boxed{\begin{array}{cc} 0 & 1 \end{array}} \; J = \boxed{\begin{array}{c} 0 \\ 1 \end{array}}$$

and let N' be determined by

$$M' = \boxed{\begin{array}{c} a \\ b \\ c \end{array}} \; J' = \boxed{\begin{array}{ccc} a & b & c \end{array}}$$

The product antilattice $N \times N'$ has generating matrices:

$$M'' = M \otimes M' = \boxed{\begin{array}{cc} 0a & 1a \\ 0b & 1b \\ 0c & 1c \end{array}} \; J'' = J \otimes J' = \boxed{\begin{array}{ccc} 0a & 0b & 0c \\ 1a & 1b & 1c \end{array}}$$

Clearly $J'' = M''^T$.

Regular Antilattices.

Note that both lattices N and N' were flat, however, their direct product $N'' = N \times N'$ is not flat. Nevertheless, the class of lattices that may be generated from flat lattices form an important subvariety of antilattices.

Definition

Any lattice that is flat or can be represented as a direct product of flat lattices is called *regular antilattice*.

Proposition

Any regular antilattice can be represented as a direct product of at most four flat antilattices (LL-flat, LR-flat, RL-flat, and RR-flat).

Regular antilattices form a variety.

If we add the following eight identities to the identities (1-4,9-10) for an antilattice we get the axioms for a regular antilattice:

$$(x \vee y) \wedge ((z \wedge x) \vee y) = x \vee y \tag{18}$$

$$(x \vee y) \wedge (x \vee (z \wedge y)) = x \vee y \tag{19}$$

$$((x \wedge z) \vee y) \wedge (x \vee y) = x \vee y \tag{20}$$

$$(x \vee (y \wedge z)) \wedge (x \vee y) = x \vee y \tag{21}$$

$$(x \wedge y) \vee ((z \vee x) \wedge y) = x \wedge y \tag{22}$$

$$(x \wedge y) \vee (x \wedge (z \vee y)) = x \wedge y \tag{23}$$

$$((x \vee z) \wedge y) \vee (x \wedge y) = x \wedge y \tag{24}$$

$$(x \wedge (y \vee z)) \vee (x \wedge y) = x \wedge y \tag{25}$$

Enumeration of Finite Regular Antilattices of order n.

Proposition

The number of regular antilattices of order n is equal to $\tau_4(n)$, the number of ordered factorizations of n of four terms.

Proof.

Each regular antilattice of order n is the direct product of four (possibly trivail) flat antilattices of respective ordera a, b, c, d such that $n = abcd$. $\square$

The sequence $\tau_4(n) = 1, 4, 4, 10, 4, 16, 4, 20, 10, \ldots$ is listed in OEIS under the name A007426.

Types of (regular) antilattices.

Proposition

For each quadruple (p, q, r, s), such that $pq = rs = n$ there exists a regular antilattice of order n having type (p, q, r, s).

Proposition

The number of types of antilattices of order n is given by $\sigma_0(n)^2$, where $\sigma_0(n)$ is the number of divisors of n.

Proof.

Each type (p, q, r, s) is uniquely determined by an independent choice of two divisors, say p and r of n. This seqence $1, 4, 4, 9, 4, 16, 4, 16, 9, \ldots$ is listed as A035116 is OEIS. $\square$

Finite antilattices form a pseudo-variety.

The class of finite antilattices is closed under:

- taking substructures
- taking homomorphic images (quotient structures?)
- taking (finite) direct products

Rectangular sub-bands.

Let B be a rectangular band. Any subband A of B is necessarily rectangular. Let M be a generating matrix for B and A a subband of B. Then some generating matrix M' for A can be obtained from M by dropping some rows and columns.
Any rectangular band B of type (p, q) has $(2^p - 1)(2^q - 1)$ subbands.

Subantilattices.

Determining all subantilattices seems to be difficult, in general.
Each antilattice N has two types of trivial subantilattices

- N itself is a subantilattice.
- Each singleton is an subantilattice.

Elementary antilattices.

Definition

If N and the singletons are the only subantilattices, we call N *elementary*.

Congruences and Quotient anti-lattices.

From now on we closely follow ideas from the wonderful paper:
J. Leech: Magic Squares, Finite Planes and Simple Quasilattices, Ars Combinatoria, 77 (2005) 75–96.

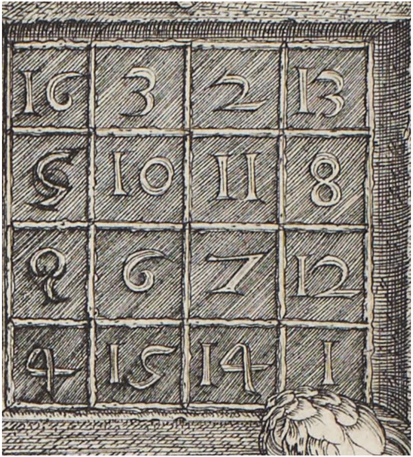
Congruences and Quotient antilattices.

Congruences in rectangular bands have simple description using generating matrices. Namely, any congruence is described by a *cartesian partition* and vice versa.

Congruences and Quotient antilattices.

Congruences in rectangular bands have simple description using generating matrices. Namely, any congruence is described by a *cartesian partition* and vice versa.
For an anitlattice, one needs compatible cartesian partitions of both generating matrices M and J. Each part in M has a corresponding part in J and together they form a sub-antilattice.

Dürer's magic square.



Dürer's magic square, as any other magic square gives rise to an antilattice.

Dürer's antilattice.

Dürer's antilattice is given by the matrices:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Dürer's antilattice and a cartesian partition.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

$$\alpha_1 : \quad [1, 2, 3, 4, 13, 14, 15, 16 | 5, 6, 7, 8, 9, 10, 11, 12]$$

Dürer's antilattice and another cartesian partition.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

$$\beta_1 : \quad [1, 2, 13, 14 | 3, 4, 15, 16 | 5, 6, 9, 10 | 7, 8, 11, 12]$$

Dürer's Congruences ...

... form a congruence lattice. The nodes are given by the following partitions given in Leech's paper:

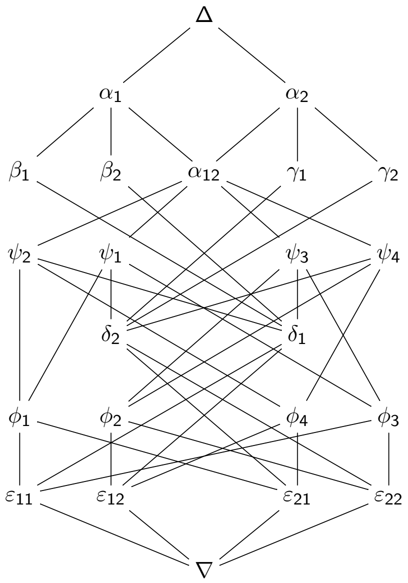
$$
\begin{array}{ll}
\alpha_1 : & [1,2,3,4,13,14,15,16|5,6,7,8,9,10,11,12] \\
\alpha_2 : & [2,4,5,8,9,12,13,16|2,3,6,7,10,11,14,15] \\
\alpha_{12} : & [1,4,13,16|2,3,14,15|5,8,9,12|6,7,10,11] \\
\beta_1 : & [1,2,13,14|3,4,15,16|5,6,9,10|7,8,11,12] \\
\beta_2 : & [1,3,13,15|2,4,14,16|5,7,9,11|6,8,10,12] \\
\gamma_1 : & [1,4,5,8|2,3,6,7|9,12,13,16|10,11,14,15] \\
\gamma_2 : & [1,4,9,12|2,3,10,11|5,8,13,16|6,7,14,15] \\
\delta_1 : & [1,13|2,14|3,15|4,16|5,9|6,10|7,11|8,12] \\
\delta_2 : & [1,4|2,3|5,8|6,7|9,12|10,11|13,16|14,15] \\
\epsilon_{11} : & [1,13|2,14|3,15|4,16|5|9|6|10|7|11|8|12] \\
\epsilon_{12} : & [1|13|2|14|3|15|4|16|5,9|6,10|7,11|8,12] \\
\epsilon_{21} : & [1,4|2|3|5,8|6|7|9,12|10|11|13,16|14|15] \\
\epsilon_{22} : & [1|4|2,3|5|8|6,7|9|12|10,11|13|16|14,15]
\end{array}
$$

Plus 8 more congruences.

together with the following partitions:

$$
\begin{aligned}
\psi_1 &: \quad [1,4,13,16|2,3,14,15|5,8|6,7|9,12|10,11] \\
\psi_2 &: \quad [1,4,13,16|2,14|3,15|5,8,9,12|6,10|7,11] \\
\psi_3 &: \quad [1,13|2,3,14,15|4,16|5,9|6,7,10,11|8,12] \\
\psi_4 &: \quad [1,4|2,3|5,8,9,12|6,7,10,11|13,16|14,15] \\
\phi_1 &: \quad [1,4,13,16|2,14|3,15|5,8|6|7|9,12|10|11] \\
\phi_2 &: \quad [1|2,3|4|5,9|6,7,10,11|8,12|13|14,15|16] \\
\phi_3 &: \quad [1,13|2,3,14,15|4,16|5|6,7|8|9|10,11|12] \\
\phi_4 &: \quad [1,4|2|3|5,8,9,12|6,10|7,11|13,16|14|15]
\end{aligned}
$$

Dürer's Congruences Lattice.



Simple antilattices.

An antilattice N is *simple* if its congruence lattice only has two elements Δ and ∇.
This definition carries over verbatim to quasilattices.

Simple antilattices.

An antilattice N is *simple* if its congruence lattice only has two elements Δ and ∇.
This definition carries over verbatim to quasilattices.

Theorem (Leech)

A simple quasilattice is either a lattice or an antilattice.

Product of antilattices is never simple.

Proposition

Let N be an antilattice that is a product of two non-trivial antilattices. Then N is not simple.

Corollary

A regular antilattice N is simple if and only if $|N| \leq 2$.

Elementary antilattices are simple.

Proposition

Every elementary antilattice is simple.

Odd antilattices.

Let N be an antilattice. We say that N in *odd* if it has no subalgebra with 2 elements.

Proposition

An elementary antilattice N of order $n, n > 2$ is odd.

Simple antilattices need not be odd or elementary.

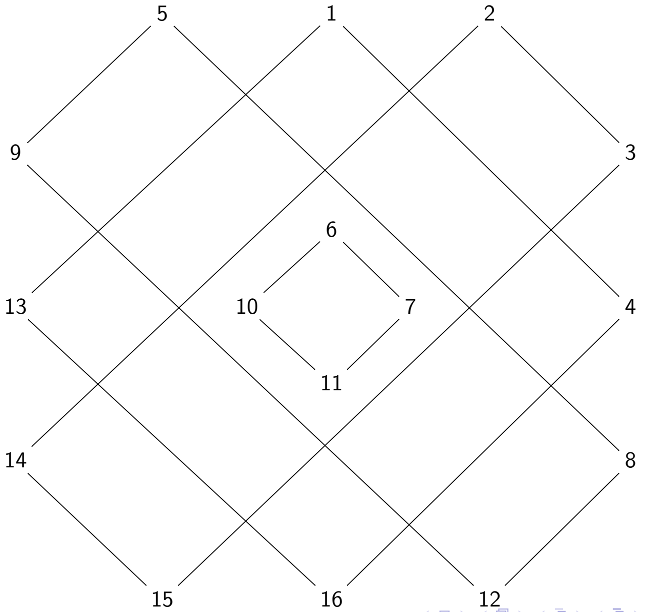
Proposition

There exist simple antilattices that are neither elementary nor odd.

The anti-diagonal graph of an antilattice.

Let N be an antilattice and x and y two of its elements.
We say that $x \sim_{hh} y$ if x and y are in the same row both in M and in J.
We say that $x \sim_{vv} y$ if x and y are in the same column both in M and in J.
We say that $x \sim_{hv} y$ if x and y are in the same row in M and in the same column in J.
We say that $x \sim_{vh} y$ if x and y are in the same column in M and in the same row in J.
Finally, let $x \sim y$ if any of the above is true. These symmetric relations define an edge-colored graph $G = (N, \sim)$ where the four colors are: hh, vv, hv, vh.

Antidiagonal graph of the Dürer's antilattice.



Example of a simple antilattice that is not elementary.

For instance, the following antilattice is simple but not elementary.

0	3	2
1	4	5
6	7	8

0	1	7
3	4	2
8	6	5

Example of a simple antilattice that is not elementary.

For instance, the following antilattice is simple but not elementary.

$$\boxed{\begin{array}{ccc} 0 & 3 & 2 \\ 1 & 4 & 5 \\ 6 & 7 & 8 \end{array}} \qquad \boxed{\begin{array}{ccc} 0 & 1 & 7 \\ 3 & 4 & 2 \\ 8 & 6 & 5 \end{array}}$$

This follows from the fact that it is not odd.

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For instance, the following antilattice is simple but not elementary.

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1	4	5
6	7	8

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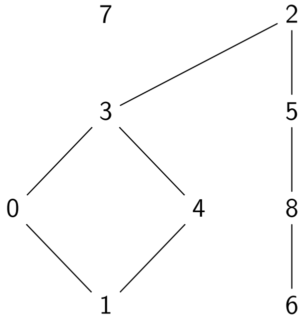

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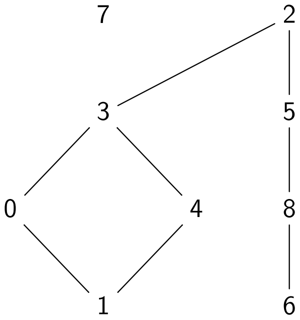

Since no automorphism of its antidiagonal graph extends to a antilattice automorphism, its automorphism group is trivial. (The antilattice is rigid.)

16 playing cards.

On the right we see a 4×4 square composed of 16 playing cards from the deck of Austrian Tarock of four different suits ($\spadesuit, \heartsuit, \diamondsuit, \clubsuit$) and four different ranks: (King (K), Queen (Q), Cavalier (C), Jack (J)).
Below we see the decomposition of this array into two arrays: one for suits and the other one for ranks.

$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$	K	Q	C	J
$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$	K	Q	C	J
$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$	K	Q	C	J
$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$	K	Q	C	J

Graeco-Latin Squares.

On the right we see an example of Graeco-Latin square. It can be also seen decomposed into a pair of orthogonal Latin squares.

$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$	K	Q	C	J
$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$	C	J	K	Q
$\heartsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$	J	C	Q	K
$\diamondsuit$	$\clubsuit$	$\spadesuit$	$\heartsuit$	Q	K	J	C

Graeco-Latin encoding.

By encoding $\spadesuit = 0, \heartsuit = 1, \diamondsuit = 2, \clubsuit = 3$ and $K = 0, Q = 1, C = 2, J = 3$ the cards on the right can be represented as the array of pairs of numbers that can be interpreted as numbers written in base 4. By adding a 1 to each number the semi-magic encoding is completed.

00	01	02	03	0	1	2	3
10	11	12	13	4	5	6	7
20	21	22	23	8	9	10	11
30	31	32	33	12	13	14	15

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Antilattice from Graeco-Latin encoding.

In this way each Graeco-Latin square gives rise to an antilattice.

… and the corresponding antilattice.

$$M = \begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ \hline \end{array}$$

$$J = \begin{array}{|cccc|} \hline 1 & 6 & 11 & 16 \\ 15 & 12 & 5 & 2 \\ 8 & 3 & 14 & 9 \\ 10 & 13 & 4 & 7 \\ \hline \end{array}$$

Graeco-Latin construction.

Let N be an antilattice. We say that N is a *Graeco-Latin antilattice* if it can be described by generating matrices M and J, where M is in the canonical form and J is a semi-magic square arising from a Gaeco-Latin square or equivalently form a pair of orthogonal Latin squares.

Main Result.

Theorem

Let N be a finite anitlattice. The following are equivalent:

- *N is odd.*
- *N is Graeco-Latin.*
- *Antidiagonal graph $G(N, \sim)$ is empty (has no edges).*

. . . and some consequences.

Corollary

Each elementary antilattice is square.

…and some consequences.

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Odd antilattice of order n exists if and only if $n = k^2, k > 2$, except for $n = 36$.

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Elementary antilattice of order $n > 2$ exists only if $n = k^2$. Elementary antilattice of order $n = 36$ does not exist.

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Elementary antilattice of order $n > 2$ exists only if $n = k^2$. Elementary antilattice of order $n = 36$ does not exist.

Corollary

Every subantilattice of an odd antilattice is odd and in particular, it is square.

...and some consequences.

Corollary

Each elementary antilattice is square.

Corollary

Odd antilattice of order n *exists if and only if* $n = k^2, k > 2$*, except for* $n = 36$*.*

Corollary

Elementary antilattice of order $n > 2$ *exists only if* $n = k^2$*.*
Elementary antilattice of order $n = 36$ *does not exist.*

Corollary

Every subantilattice of an odd antilattice is odd and in particular, it is square.

Question

Is there an order $n = k^2, k \notin \{2, 6\}$ *for which no elementary antilattice exists?*

Odd antilattices that are simple.

Proposition

There exist infinitely many odd antilattices that are simple.

Proof.

Take an odd antilattice of order p^2 where p is prime. $\square$

A useful result on orthogonal Latin squares.

In 1986 K. Heinrich and L. Zhu solved the remaining even case and gave a complete answer to the question for which parameters v and n, $v > n$ there exist a pair of Graeco-Latin squares of orders v and n such that the small square is contained in the larger one.

Theorem (Heinrich and Zhu, 1986)

A pair of orthogonal Latin squares of order n can be embedded into a pair of orthogonal Latin squares of order $v > n$, if and only if $v \geq 3n$ and $v \neq 6$ and $n \neq 2, 6$.

Some consequences

Corollary

There exist infinitely many finite simple antilattices that are not elementary.

Proof.

Take v prime and $v > 3n$. $\square$

An odd antilattice of order 81 that is neither simple nor elementary.

$$M = \begin{array}{|ccc|ccc|ccc|}
\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \\
\hline
27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 \\
36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 \\
45 & 46 & 47 & 48 & 49 & 50 & 51 & 52 & 53 \\
\hline
54 & 55 & 56 & 57 & 58 & 59 & 60 & 61 & 62 \\
63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 & 71 \\
72 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 \\
\hline
\end{array}$$

$$J = \begin{array}{|ccc|ccc|ccc|}
\hline
0 & 20 & 10 & 60 & 80 & 70 & 30 & 50 & 40 \\
19 & 9 & 2 & 79 & 69 & 62 & 49 & 39 & 32 \\
11 & 1 & 18 & 71 & 61 & 78 & 41 & 31 & 48 \\
\hline
57 & 77 & 67 & 27 & 47 & 37 & 6 & 26 & 16 \\
76 & 66 & 59 & 46 & 36 & 29 & 25 & 15 & 8 \\
68 & 58 & 75 & 38 & 28 & 45 & 17 & 7 & 24 \\
\hline
33 & 53 & 43 & 3 & 23 & 13 & 54 & 74 & 64 \\
52 & 42 & 35 & 22 & 12 & 5 & 73 & 63 & 56 \\
44 & 34 & 51 & 14 & 4 & 21 & 65 & 55 & 72 \\
\hline
\end{array}$$

Another consequence.

Corollary

Let N be an odd antilattice of order n. If $n < 81$ then N is elementary.

Thank you!