Integrality of some Cayley graphs

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based on a joint work with
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Integral graphs

- Let Γ be a graph, $A(\Gamma)$ its adjacency matrix.
  The spectrum $\mathrm{Spec}(\Gamma)$ of Γ is the spectrum of $A(\Gamma)$,
  i. e. the set of its eigenvalues.
- A graph Γ is said to be integral, if $\mathrm{Spec}(\Gamma) \subseteq \mathbb{Z}$.

Why the integrality is useful?

- A regular graph is a graph where each vertex has the same degree, i.e. the number of neighbors.
- A regular graph with vertices of degree k is called a k-regular graph.
- If a graph Γ is k-regular then
$$\mathrm{Spec}(\Gamma) \subseteq [-k, k].$$
- If a graph Γ is k-regular and integral then
$$\mathrm{Spec}(\Gamma) \subseteq \{0, \pm, 1, \pm 2, \dots, \pm k\}.$$
- Thus, if a regular graph is integral, then we can efficiently calculate its spectrum.

F. Harary, A. J. Schwenk, 1974

Which graphs have integral spectra?

O. Ahmadi, N. Alon, I. F. Blake, and I. E. Shparlinski; 2009

- Most graphs have nonintegral eigenvalues.
- More precisely, the probability of a labeled graph on n vertices to be integral is at most $2^{-n/400}$ for a sufficiently large n.
- We believe our bound is far from being tight and the number of integral graphs is substantially smaller.

F. C. Bussemaker, D. Cvetkovič, 1976; Schwenk, 1978

There are exactly 13 connected, cubic (i.e. 3-regular), integral graphs.

- Let G be a finite group, S a nonempty subset of $G \setminus \{1\}$, containing with every element its inverse, i. e. $S = S^{-1} = \{s^{-1} \mid s \in S\}$.
- Then we say that S is symmetric.
- The Cayley graph of a group G associated with a symmetric subset S is an undirected graph $\mathrm{Cay}(G, S)$ with the vertex set identified with G, and vertices $x, y \in G$ are adjacent if and only if there exists $s \in S$, such that $y = xs$.
- $\mathrm{Cay}(G, S)$ is k-regular, where $k = |S|$.
- $\mathrm{Cay}(G, S)$ is connected, iff $G = \langle S \rangle$.

A. Abdollahi, E. Vatandoost, 2009

Which Cayley graphs have integral spectra?

- W. So (Cayley graphs over cyclic groups);
- W. Klotz, T. Sanders (Cayley graphs over abelian groups);
- L. Lu, Q. Huang, X. Huang (Cayley graphs over dihedral groups).

Integrality of the star graphs

The symmetric group Sym_n of degree n

- is generated by the set of all transpositions: $\mathrm{Sym}_n = \langle (ij) \mid 1 \leqslant i < j \leqslant n \rangle$;
- is generated by the set $T = \{(1i) \mid i = 2, \dots, n\}$ of $n-1$ transpositions;
- can not be generated by any set of transpositions of cardinality less than $n-1$.

A star graph is $\mathrm{Cay}(\mathrm{Sym}_n, T)$, where $T = \{(1i) \mid i = 2, \dots, n\}$.

Conjecture (A. Abdollahi and E. Vatandoost, 2009)

Every star graph is integral and, if $n > 3$, then its spectrum contains all integers in the range from $-(n-1)$ up to $n-1$.

R. Krakowski and B. Mohar, 2012
G. Chapuy and V. Feray, 2012

This Conjecture is true.

J. Friedman, 2002

Suppose, T is a set of $n-1$ transpositions in Sym_n such that $\mathrm{Sym}_n = \langle T \rangle$ and $\Gamma = \mathrm{Cay}(\mathrm{Sym}_n, T)$ is integral. Then Γ is the Star graph, i. e. up to conjugation, $T = \{(1i) \mid i = 2, \dots, n\}$.

The multiplicities of the eigenvalues of the Star graph was studied by R. Krakowski, B. Mohar, G. Chapuy, V. Feray, S. V. Avgustinovich, E. V. Konstantinova, E. M. Khomyakova.

Using the integrality of the star graphs, the following two statements are obtained.

E. V. Konstantinova and D. V. Lytkina, 2018

Let T be the set all transpositions in Sym_n, i. e. $T = \{(ij) \mid 1 \leqslant i < j \leqslant n\}$. Then $\mathrm{Cay}(\mathrm{Sym}_n, T)$ is integral.

E. V. Konstantinova and D. V. Lytkina, 2018

Let $S = \{(1ij) \mid 2 \leqslant i, j \leqslant n, i \neq j\}$. Then $\Gamma = \mathrm{Cay}(\mathrm{Alt}_n, S)$ is integral and $\mathrm{Spec}(\Gamma) = \{k^2 - (n-1) \mid k = 0, \dots, n-1\}$.

E. V. Konstantinova and D. V. Lytkina, 2018

Let $T = \{(ij) \mid 1 \leqslant i < j \leqslant n\}$. Then $\mathsf{Cay}(\mathsf{Sym}_n, T)$ is integral.

The set of all transpositions is a normal subset of Sym_n.
A set $S \subseteq G$ is said to be normal, if $\forall g \in G \quad s \in S \Rightarrow s^g \in S$.

D. V. Lytkina, Kourovka Notebook, Problem 19.50(a)

Is it true that if $S \subseteq G$ is a normal set of involutions (i. e. elements of order 2) then $\mathsf{Cay}(G, S)$ is integral?

In the same peper, Konstantinova and Lytkina have solved this problem in the affirmative for the case where G is nilpotent, i. e. G is the direct product of its Sylow subgroups.

E. V. Konstantinova and D. V. Lytkina, 2018

Let $S = \{(1ij) \mid 2 \leqslant i, j \leqslant n, i \neq j\}$. Then $\mathsf{Cay}(\mathsf{Alt}_n, S)$ is integral.

D. V. Lytkina, Kourovka Notebook, Problem 19.50(b)

Let $R = \{(12i) \mid 3 \leqslant i \leqslant n\}$ and $S = R \cup R^{-1}$. Is it true that $\mathsf{Cay}(\mathsf{Alt}_n, S)$ is integral?

If $x, y \in G$, then $\langle x \rangle = \langle y \rangle$ iff $y = x^k$ where $(k, |x|) = 1$.
There are exactly $\phi(|x|)$ elements $y \in G$ such that $\langle x \rangle = \langle y \rangle$.
Here ϕ is the Euler function: $\phi(m)$ is the number of positive integers k such that $k \leqslant m$ and $(k, m) = 1$.
If $m = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ is the canonical prime decomposition, then

$$\phi(m) = p_1^{\alpha_1 - 1}(p_1 - 1) \dots p_t^{\alpha_t - 1}(p_t - 1).$$

Definition

We say that a subset S of a group G is an Euler set,
if S contains all generators $\langle s \rangle$ for every $s \in S$,
i. e. if $s \in S$, then
$\{x \in G \mid \langle x \rangle = \langle s \rangle\} = \{s^k \mid (k, |s|) = 1, \quad 0 \leqslant k \leqslant |s| - 1\} \subseteq S$.

Every Euler set is symmetric, since $\langle s \rangle = \langle s^{-1} \rangle$.

If $S \subseteq G \setminus \{1\}$ is symmetric and $\{|s| \mid s \in S\} \subseteq \{2, 3, 4, 6\}$ (equivalently, $\phi(|s|) \leqslant 2$ for all $s \in S$), then S is an Euler set.

E. V. Konstantinova and D. V. Lytkina, 2018

If G is nilpotent and $S \subseteq G$ is a normal Euler set, then $\mathrm{Cay}(G, S)$ is integral.

Theorem (W. Guo, D. V. Lytkina, V. D. Mazurov, R. 2018)

If $S \subseteq G$ is a normal Euler set, then $\mathrm{Cay}(G, S)$ is integral.

Corollary 1

If $S \subseteq G$ is normal, symmetric, and $|s| \in \{2, 3, 4, 6\}$ for all $s \in S$, then $\mathrm{Cay}(G, S)$ is integral.

Corollary 2

If $S \subseteq G$ is a normal set of involutions, then $\mathrm{Cay}(G, S)$ is integral.

This statement was independently proved by A. Abdollahi.

Theorem (W. Guo, D. V. Lytkina, V. D. Mazurov, R. 2018)

If $S \subseteq G$ is a normal Euler set, then $\mathsf{Cay}(G, S)$ is integral.

Sketch of the proof

- For every $S \subseteq G$, denote the sum $\sum_{s \in S} s$ in $\mathbb{C}G$ by $\overline{S}$.
- The adjacency matrix of $\Gamma = \mathsf{Cay}(G, S)$ coincides with the matrix of the linear transformation $a \mapsto a\overline{S}$ of $\mathbb{C}G$ in the basis $\{g \mid g \in G\}$.
- The Wedderburn theorem implies that Spec(Γ) is the union of the spectra of the transformations $\overline{S}_\chi : v \mapsto v\overline{S}$ of the irreducible modules V_χ, where $\chi \in \mathsf{Irr}(G)$.
- Since $\overline{S} \in Z(\mathbb{C}G)$ for normal S, the Schur lemma implies that the matrix of $\overline{S}_\chi$ is scalar.

- The spectrum of $\overline{S}_\chi$ is $\{\lambda_\chi\}$, where

$$\lambda_\chi = \sum_{i=1}^{t} \omega_\chi(\overline{K}_i), \qquad (*)$$

$K_1 = x_1^G, \ldots, K_t = x_t^G$ are the conjugacy classes of G such that $S = K_1 \sqcup \cdots \sqcup K_t$ and

$$\omega_\chi(\overline{K}_i) = \frac{\chi(x_i)|K_i|}{\chi(1)}.$$

- It is well-known that $\omega_\chi(\overline{K}_i)$ is an algebraic integer and lies in $\mathbb{Q}(\zeta)$, where ζ is an m-th primitive root of 1 for $m = \mathrm{lcm}(|s| \mid s \in S)$.
- Since S is an Euler set, it is easy to show that every automorphism of $\mathbb{Q}(\zeta)$ permutes the summands in $(*)$.
- Thus, $\lambda_\chi \in \mathbb{Q}$ and $\lambda_\chi \in \mathbb{Z}$.

Corollary 3

Let $R \subseteq G$ be a normal Euler subset and $H \leqslant G$.
Put $S := R \setminus (R \cap H)$. Then $\mathsf{Cay}(G, S)$ is integral.

Let $G := \mathsf{Sym}_n$,
$R := \{(ij) \mid 1 \leqslant i < j \leqslant n\}$, and
$H \cong \mathsf{Sym}_{n-1}$ the stabilizer of 1.
Then $S := \{(1i) \mid i = 2, \ldots, n\}$ coincides with $R \setminus (R \cap H)$ and $\mathsf{Cay}(G, S)$ is integral.

Corollary 4

The graph $\mathsf{Cay}(\mathsf{Sym}_n, S)$ is integral for $S = \{(1i) \mid i = 2, \ldots, n\}$.

Corollary 3 could be a source of new integral Cayley graphs (groups of 3-transpositions etc).

Corollary 3′

Let $R \subseteq G$ be a normal Euler set, and let $H_1, \ldots H_n \leqslant G$ such that $[H_i, H_j] = 1$ for $i \neq j$. Put $S := R \setminus \bigcup_{i=1}^{n} (R \cap H_i)$.
Then $\mathsf{Cay}(G, S)$ is integral.

Corollary 3

Let $R \subseteq G$ be a normal Euler set, and let $H \leqslant G$. Put $S := R \setminus (R \cap H)$. Then $\mathsf{Cay}(G, S)$ is integral.

Corollary 5

$\mathsf{Cay}(\mathsf{Alt}_n, S)$ is integral for $S = \{(12i)^{\pm 1} \mid i = 3, \ldots, n\}$.

This statement was independently proved by M. Muzychuk.

$\mathsf{Spec}(\mathsf{Cay}(\mathsf{Alt}_n, S)) = \mathsf{Spec}(\mathsf{Cay}(\mathsf{Sym}_n, S))$ for $S \subseteq \mathsf{Alt}_n$.
Let $S := \{(12i)^{\pm 1} \mid i = 3, \ldots, n\}$,
$G := \mathsf{Sym}_n$, $R := \{(ij) \mid 1 \leqslant i < j \leqslant n\}$, $H \cong \mathsf{Sym}_2 \times \mathsf{Sym}_{n-2}$ the stabilizer of $\{1, 2\}$, and let $T := \{(1i), (2i) \mid i = 3, \ldots, n\}$.
Then $T = R \setminus (R \cap H)$ and $\mathsf{Spec}(\mathsf{Cay}(G, T)) \subseteq \mathbb{Z}$.
Consider $a := \sum_{s \in S} s$, $b := \sum_{t \in T} t$, and $c := (12)$ in $\mathbb{C}G$.
Then $a = bc = cb$. There is a basis of $\mathbb{C}G$ consisting of common eigenvectors of a, b, and c. If $\alpha \in \mathsf{Spec}(\mathsf{Cay}(G, S)) = \mathsf{Spec}(a)$, then $\alpha = \beta\gamma$ for some $\beta \in \mathsf{Spec}(b) = \mathsf{Spec}(\mathsf{Cay}(G, T)) \subseteq \mathbb{Z}$ and $\gamma \in \mathsf{Spec}(c) \subseteq \{1, -1\}$. Therefore,
$\mathsf{Spec}(\mathsf{Cay}(\mathsf{Alt}_n, S)) = \mathsf{Spec}(\mathsf{Cay}(G, S)) \subseteq \mathbb{Z}$.

Open problems

Problem 1

Let $S \subseteq G \setminus \{1\}$ be a symmetric normal set. Assume, $\mathsf{Cay}(G, S)$ is integral. Is it true that S is an Euler set?

Problem 2

Calculate the eigenvalues and their multiplicities for $\Gamma = \mathsf{Cay}(\mathsf{Alt}_n, S)$, where $S = \{(12i)^{\pm 1} \mid i = 3, \ldots, n\}$.

A. Yu. Ovcharenko calculated $\mathsf{Spec}(\Gamma)$ for $n \leqslant 8$.

Problem 3

Calculate the eigenvalues and their multiplicities for $\Gamma = \mathsf{Cay}(\mathsf{Sym}_n, S)$, where $S = \{(ij) \mid 1 \leqslant i < j \leqslant n\}$.

A. Yu. Ovcharenko calculated $\mathsf{Spec}(\Gamma)$ for $n \leqslant 6$.
In general case, $\mathsf{Spec}(\Gamma) = \left\{ \frac{n(n-1)}{2} \frac{a_\mu - b_\mu}{a_\mu + b_\mu} \mid \mu \vdash n \right\}$,
where a_μ and b_μ are, respectively, the numbers of paths in the Young graph from $(2) \vdash 2$ and $(1^2) \vdash 2$ to $\mu \vdash n$.

References

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Thank you for your attention!