ON 2-CLOSURES OF PRIMITIVE SOLVABLE PERMUTATION GROUPS

Evgeny Vdovin[1]

[1]Sobolev Institute of mathematics SB RAS

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Ω is a finite set of size n and G is a subgroup of $Sym(\Omega)$.
G acts coordinatewisely on Ω^k: $(\omega_1, \ldots, \omega_k)^g = (\omega_1^g, \ldots, \omega_k^g)$.
$Orb_k(G)$ is the set of orbits under this action.
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$$G^{(1)} \geqslant G^{(2)} \geqslant \ldots \geqslant G^{(n)} = G.$$

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If G is not primitive, then G is contained in $Sym_k \wr Sym_m$ with $km = n$ and again one can try to use induction in this case.
So we start with primitive solvable groups

Now assume that G is a primitive solvable linear group. Then G contains a normal regular elementary abelian subgroup V of order p^k. So we can identify Ω and V, in particular $n = p^k$.

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Denote by L the stabilizer of 0 in G. Then $G = V \rtimes L$ and the action of L on $V \setminus \{0\}$ is permutationally equivalent to the action of L on $\Omega \setminus \{0\}$.
Thus

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$$G \leqslant AGL(V) = AGL_k(p) = V \rtimes GL_k(p).$$

(Praeger, Saxl, 1992)

If $G = V \rtimes L$ is a primitive solvable permutation group, then $G^{(2)} \leqslant AGL(V) = V \rtimes GL(V)$.

So $G^{(2)} = V \rtimes H$, where $H \leqslant GL_k(V)$.

Lemma

Let $G \leqslant Sym(\Omega)$ be transitive, $L = Stab_\omega(G) = G_\omega$ and $\Delta \in Orb_2(G)$. Then $\Gamma = \{\alpha \in \Omega \mid (\omega, \alpha) \in \Delta\} \in Orb_1(L)$. In particular there exists a bijection $\varphi : Orb_2(G) \to Orb_1(L)$.

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If L is linearly imprimitive and fixes a decomposition $V = V_1 \otimes \ldots \otimes V_m$, then $G = V \rtimes L \leqslant Sym_{p^{k/m}} \wr Sym_m$ acting on Ω by product action and again one may hope to use an induction in this case.
So we are assuming that G is a primitive solvable group, so that $G = V \rtimes L$, and, moreover L is linearly primitive subgroup of $GL(V)$ and L does not act transitively on $V \setminus \{0\}$.

D.Suprunenko, 60's

Let L be a maximal finite solvable linearly primitive subgroup of $GL_k(p)$. Then K has a series $1 < A \leqslant B \leqslant C \leqslant L$ of normal subgroups such that the following statements hold:

1 $A \cong Z_{p^a-1}$ and $Span(A) = GF(p^a)$, where a is a divisor of k.

2 $|B : A| = e^2$, where $e = k/a$; moreover, each prime divisor of e divides $p^a - 1$.

(3) C/B can be identified with a completely reducible subgroup of the group $\prod_{i=1}^m Sp(2n_i, p_i)$ where n_i and p_i are such that $e = \prod_{i=1}^m p_i^{n_i}$.

(4) L/C is isomorphic to a subgroup of $Aut(GF(p^a))$; in particular, $|K : C|$ divides a.

Ponomarenko and V, unpublished

Let $G \leqslant AGL_k(p)$ be a primitive solvable linearly primitive permutation group. Suppose that $e \geqslant 9$. Then the group $G^{(2)}$ can be found in time $poly(n)$, where $n = p^k$. Moreover, if G is not 2-transitive, then either $G^{(2)}$ is solvable or $k \leq k_0$ for a constant k_0.

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Conjecture

If G is a solvable subgroup of $Sym(\Omega)$, then all composition factors of $G^{(2)}$ are either abelian, or are isomorphic to alternating groups in natural action.

In general the answer to the conjecture is negative. For example, if we take a subgroup G of $A\Gamma L_1(2^6) = V \rtimes \langle \omega \rangle \rtimes \langle \alpha \rangle \leqslant AGL_6(2)$ generated by V, ω^3, α, then $G^{(2)}$ has a nonabelian composition factor isomorphic to $SL_3(2)$.

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Conjecture

If G is a solvable subgroup of $\mathrm{Sym}(\Omega)$, then all comosition factors of $G^{(2)}$ are either abelian, or are isomorphic to alternating groups in natural action, or of bounded by some absolute constant order.

THANK YOU!