Alphabet lifting construction of equitable partitions of Hamming graphs

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Hamming graphs

The Hamming graph $H(n,q)$ is a graph whose vertices are all words of length n over the alphabet $\{0,1,\ldots,q-1\}$. Two vertices are adjacent if and only if they differ in exactly one coordinate position.

Eigenvalues of $H(n,q)$ are defined by the set $\{\lambda_i(n,q) = n(q-1) - qi, i = 0,1,\ldots,n\}$. Corresponding eigenspaces $V_i(n,q)$ have multiplicities $mult(\lambda_i) = \binom{n}{i}(q-1)^i$.

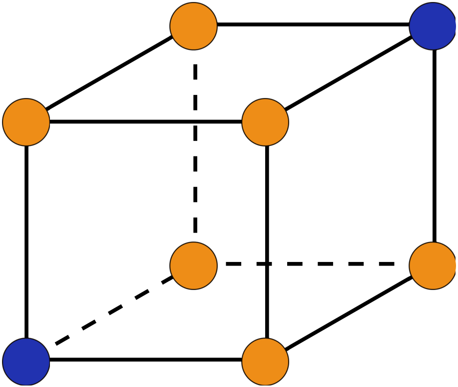
Equitable partitions

Let $G = (V, E)$ be an undirected graph. A partition $(C_1, \ldots, C_t)$ of the set V is an equitable partition if for all $i, j \in \{1, \ldots, t\}$ any vertex of C_i has exactly m_{ij} neighbors in C_j. A matrix $M_{t\times t} = (m_{ij})$ is called the quotient matrix (or the matrix of parameters) of this partition.

Equitable partition, perfect coloring, partition design, regular partition, ...

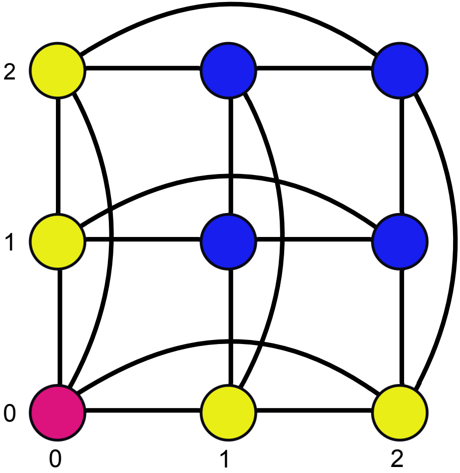
Examples

The equitable partition of $H(3,2)$ with the quotient matrix $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$.



Examples

Equitable partition of $H(2,3)$ with the quotient matrix $\begin{pmatrix} 0 & 4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$.



Eigenvalues of equitable partitions

Suppose we have a graph G with an adjacency matrix A and some equitable partition of this graph G with a quotient matrix M. It is known that

$$Sp(M) \subseteq Sp(A)[1]$$

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[1] D. M. Cvetkovic, M. Doob, H. Sachs, Spectra of graphs, Academic Press, New York, London, (1980)

Eigenspaces

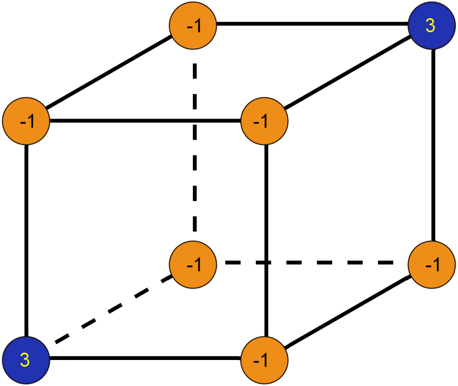
Let $G = (V, E)$ be an arbitrary nonoriented graph with an adjacency matrix M. A real-valued function $f : V \to \mathbb{R}$ is called a λ-eigenfunction of G if the following equality holds for any $x \in V$:

$$\lambda f(x) = \sum_{y \in V:(x,y) \in E} f(y).$$

In other words, if for the vector of values $\overline{f}$ the following holds:

$$M\overline{f} = \lambda\overline{f}.$$

Example of an eigenfunction



The eigenfunction corresponding to the equitable partition of $H(3,2)$ with the quotient matrix $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$.

Eigenfunctions and equitable 2-partitions

Let us have a partition (C_1, C_2) of $H(n, q)$ with a quotient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Consider a function $f : V(H(n, q)) \to \mathbb{R}$ such that

$$f = \begin{cases} b, \text{ if } x \in C_1, \\ -c, \text{ if } x \in C_2. \end{cases}$$

Then f is an $(a - c)$-eigenfunction of $H(n, q)$.

Main problem

1. Classification of all equitable partitions of $H(n, q)$ with given quotient matrix M.
2. Classification of all quotient matrices M for which some equitable partition exist.
3. Classification of all quotient matrices M for which some equitable 2-partition exist.

Equitable partitions of $H(n, q)$

Equitable partitions of the Hamming graph contain

1. Perfect codes [2] [3]
2. Completely regular codes [4]

[2] Solov'eva F.I. On perfect binary codes. Discrete Appl. Math. 156(9), 1488–1498 (2008)

[3] O. Heden. A survey of perfect codes. Advances in Mathematics of Communications, 2(2), 223-247 (2008)

[4] J. Borges, J. Rifa, V. A. Zinoviev. On Completely Regular Codes, arXiv:1703.08684 (2017)

General constructions

D.G. Fon-Der-Flaass proposed several iterative constructions of equitable partitions of $H(n,2)$ [5], which can be easily generalized for the case $q > 2$.

Let $C = (C_1, C_2, \dots C_t)$ be an equitable partition of $H(n,q)$ with a quotient matrix M. Define the partition of vertices $(C'_1, \dots, C'_t)$ of the graph $H(n+1,q)$ as follows:

$$\forall i \in \{1, \dots, t\} \; \forall x = (x_1, x_2, \dots, x_n, x_{n+1}) \in H(n+1, q)$$

$$(x_1, x_2, \dots, x_n, x_{n+1}) \in C'_i \iff (x_1, \dots, x_n) \in C_i.$$

[5] D.G. Fon-Der-Flaass, Perfect 2-colorings of a hypercube. Siberian Mathematical Journal 48(4) 740-745 (2008).

General constructions

Let $C = (C_1, C_2, \ldots C_t)$ be an equitable partition of $H(n,q)$ with a quotient matrix M. Define the partition of vertices $(C_1', \ldots, C_t')$ of the graph $H(n \cdot k, q)$ as follows: $\forall i \in \{1, \ldots, t\}$

$$\forall x = (x_{1,1}, x_{1,2}, \ldots, x_{1,k}, x_{2,1}, x_{2,2}, \ldots, x_{2,k}, \ldots, x_{n,k}) \in H(n \cdot k, q)$$

$$x \in C_i' \iff (\sum_{j=1}^{k} x_{1,j} \bmod q, \ldots, \sum_{j=1}^{k} x_{n,j} \bmod q) \in C_i.$$

General constructions

1. An equitable partition of $H(n, q)$ with a quotient matrix $M \longrightarrow$ an equitable partition of $H(n+1, q)$ with a quotient matrix $M + (q-1)E$.

2. An equitable partition of $H(n, q)$ with a quotient matrix $M \longrightarrow$ an equitable partition of $H(n \cdot k, q)$ with a quotient matrix $k \cdot M$.

Alphabet lifting construction

Theorem (Vorob'ev, 2018).
Let $(C_1, \dots, C_t)$ be an equitable partition of the graph $H(n, q_1)$ with the matrix of parameters M. Define the partition of vertices $(C'_1, \dots, C'_t)$ of the graph $H(n, q_1 q_2)$ as follows:

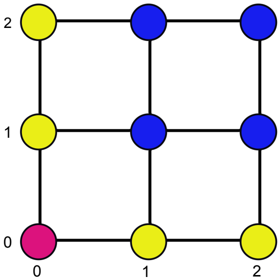
$$\forall i \in \{1, \dots, t\}\ \forall x = (x_1, x_2, \dots, x_n) \in H(n, q_1 q_2)$$

$$(x_1, x_2, \dots, x_n) \in C'_i \iff (x_1 \bmod q_1, \dots, x_n \bmod q_1) \in C_i.$$

Then $(C'_1, \dots, C'_t)$ is an equitable partition with the quotient matrix $q_2 M + n(q_2 - 1)E$, where E is the identity matrix of order t.

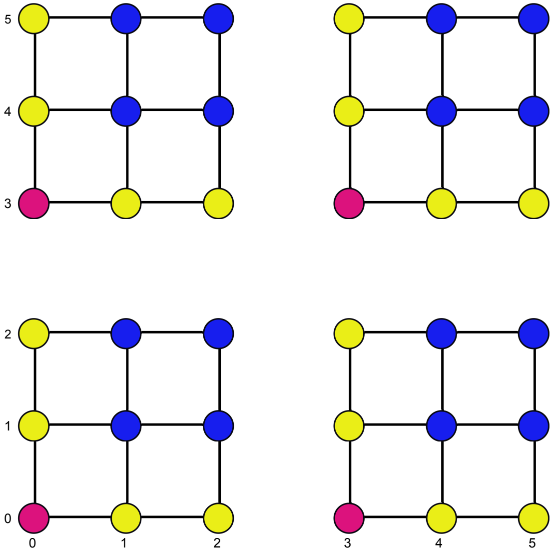
Example

A partition (C_1, C_2, C_3) of $H(2,3)$ with quotient matrix $\begin{pmatrix} 0 & 4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$.

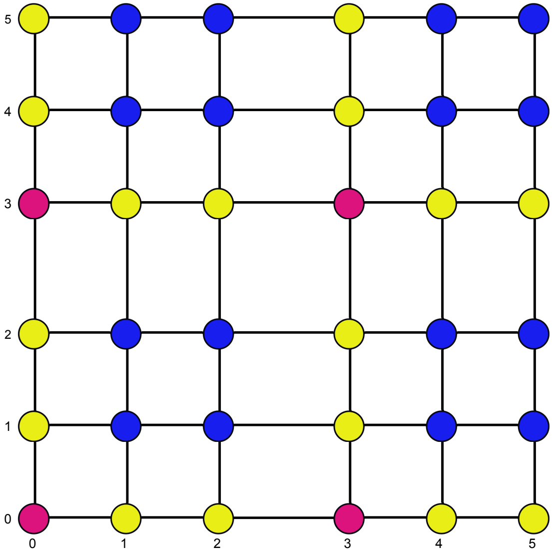


Using the construction we obtain a partition (C_1', C_2', C_3') of $H(2,6)$ with a quotient matrix $\begin{pmatrix} 2 & 8 & 0 \\ 2 & 4 & 4 \\ 0 & 4 & 6 \end{pmatrix}$.

Example



Example



Construction for eigenfunctions

Let f be a $\lambda_i(n, q_1)$-eigenfunction of the graph $H(n, q_1)$. Define the function f' in the graph $H(n, q_1 q_2)$ as follows:

$$\forall x = (x_1, x_2, \dots, x_n) \in V(H(n, q_1 q_2))$$

$$f'(x_1, x_2, \dots, x_n) = f(x_1 \bmod q_1, \dots, x_n \bmod q_1).$$

Then f' is a $\lambda_i(n, q_1 q_2)$-eigenfunction of $H(n, q_1 q_2)$.

Equitable 2-partitions of $H(n, q)$

Let $C = (C_1, C_2)$ be an equitable 2-partition of $H(n, q)$ with a quotient matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

There are several necessary conditions:

1. $a + b = c + d = n(q - 1)$
2. $\frac{b+c}{q} \in \{1, 2, \ldots, n\}$
3. $\frac{c}{b+c} q^n \in \mathbb{N}$

Equitable 2-partitions of $H(n,2)$

Theorem (Fon-Der-Flaass) [6] Let us have $b, c \in \mathbb{N}$ such that $\frac{b+c}{gcd(b,c)} = 2^t$ for some integer t. Then there exist $a_0 \in \mathbb{N}$ such that $\forall a \geq a_0$ there exist a perfect 2-coloring of $H(a+b,2)$ with a quotient matrix

$$\begin{pmatrix} a & b \\ c & a+b-c \end{pmatrix}.$$

The bound on correlation immunity[7] : if $b \neq c$ then $a - c \geq \frac{n}{3}$.

There is no[8] perfect 2-coloring of $H(12,2)$ with the quotient matrix

$$\begin{pmatrix} 1 & 11 \\ 5 & 7 \end{pmatrix}.$$

[6] D.G. Fon-Der-Flaass, Perfect 2-colorings of a hypercube. Siberian Mathematical Journal 48(4) 740-745 (2008).

[7] D.G.Fon-Der-Flaass, A bound on correlation immunity. Sib. Elektron. Mat. Izv. 4 133-135 (2007).

[8] D.G. Fon-Der-Flaass, Perfect colorings of the 12-cube that attain the bound on correlation immunity. Sib. Elektron. Mat. Izv. 4 (2007) 292-295.

Constructing 2-partitions

Combining the alphabet lifting construction with previously known constructions one can find equitable 2-partitions with new quotient matrices.

Corollary 1 Take arbitrary $k, m', m'', b, c \in \mathbb{N}$ and prime p such that

$$m'' < m',\ b + c = kp^{m'},\ gcd(b, c) = kp^{m''},\ gcd(k, p) = 1.$$

Let k_1 be an arbitrary natural divisor of k and s be an arbitrary natural divisor of one of the numbers $m', m' - 1, \ldots, m' - m''$. Then there exist $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ there exist an equitable 2-partition of the graph $H(n, q)$ for $q = k_1 p^s$ with the matrix of parameters

$$\begin{pmatrix} n(q-1) - b & b \\ c & n(q-1) - c \end{pmatrix}$$

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Equitable 2-partitions of $H(n, q)$ with the eigenvalue λ_2

Recently Ivan Mogilnykh and Alexander Valyuzhenich [9] obtained a description of all equitable 2-partitions of $H(n, q)$ such that the spectra of corresponding quotient matrix contains λ_2.

One of their families of equitable 2-partitions is based on the alphabet lifting construction.

[9] I. Mogilnykh, A. Valyuzhenich. On equitable 2-partitions of $H(n, q)$ with eigenvalue λ_2. Abstracts of G2R2 (2018).

Future work

1. New approaches to find all possible quotient matrices of equitable 2-partitions
2. Application of the alphabet lifting construction to known families of completely regular codes

Thank you for your attention.