An expansion property of Boolean multivariable digraphs

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G2R2, Novosibirsk, 18 August 2018

References/plan for this talk

(i) Yaokun Wu, Zeying Xu (徐泽瀛), Yinfeng Zhu (祝隐峰), Strongly connected multivariate digraphs, The Electronic Journal of Combinatorics, 24(1) (2017) #P1.47, 44 pp.

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(ii) H. Schneider, Wielandt's proof of the exponent inequality for primitive nonnegative matrices, Linear Algebra and its Applications, 353 (2002) 5–10.

(iii) P.G. Coxson, L.C. Larson, H. Schneider, Monomial patterns in the sequence $A^k b$, Linear Algebra and its Applications, 94 (1987) 89–101.

(iv) Yaokun Wu, Zeying Xu, Yinfeng Zhu, An expansion property of Boolean linear maps, The Electronic Journal of Linear Algebra, 31 (2016) 381–407.

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(iv) Yaokun Wu, Zeying Xu, Yinfeng Zhu, An expansion property of Boolean linear maps, The Electronic Journal of Linear Algebra, 31 (2016) 381–407.

(v) Zongchen Chen (陈宗晨), Yaokun Wu, An expansion property of Boolean multilinear maps, preprint.

Boolean multivariable digraphs

What are numbers and what is their meaning? – Richard Dedekind, 1888.

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- What are digraphs?

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- What are digraphs?
- One possible answer: A digraph on a vertex set V is a map from V to 2^V.

例

A digraph f on the vertex set $\{A, B, C, D\}$:

	A	B	C	D
f	{B}	{C,D}	{A}	∅

An example

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f	{B}	{C,D}	{A}	∅

The digraph f represented by a ‘more typical’ object:

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \arrow[d] \\ C \arrow[u] & D \end{tikzcd}$$

Single-variable graph theory

Let K be a set. A Boolean single-variable digraph f, also known as a topological Markov chain, on K is a map from K to 2^K.

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By linear extension, the map f induces a unique Boolean linear map M_f from 2^K to 2^K, called the Markov operator associated to f, such that

$$\mathrm{M}_f(A) = \cup_{k \in A} f(k)$$

for each $A \in 2^K$.

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The phase space of f, denoted by $\mathcal{PS}_f$, has vertex set 2^K and arc set $\{A \to \mathrm{M}_f(A) : \ A \in 2^K\}$. The structure of $\mathcal{PS}_f$ tells us the dynamical behavior of M_f.

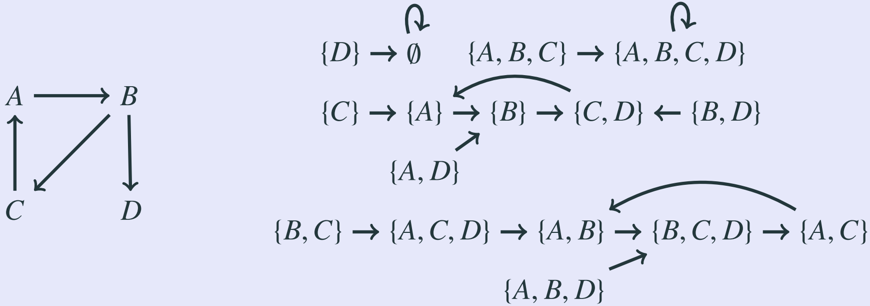

Figure 1: A digraph f on $\{A, B, C, D\}$, its De Bruijn form Γ_f and its phase space $\mathcal{PS}_f$.

The phase space consists of those limit cycles together with the transient parts (in-trees attached to the limit cycles).

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Arnold called a single-valued map a monad and suggested to study the dynamical behavior of several concrete monads, say determining the lengths of limit cycles and maximum depth of the transient part. Single-valued maps may already be difficult enough to understand.

Let K be a set and t a positive integer. A Boolean digraph f on K in t variables is a map from K^t to 2^K.

Tropical group as a 2-variable Boolean digraph

Tropical groups

A binary multi-valued operation in X: a map $X \times X \to 2^X \smallsetminus \{\varnothing\}$.

A set X with a multi-valued operation $(a, b) \mapsto a \top b$ is a commutative tropical group if

1. $\top$ is commutative;
2. $\top$ is associative;
3. X contains 0 such that $0 \top a = a$ for any $a \in X$;
4. for each $a \in X$ there exists a unique $-a \in X$ such that $0 \in a \top (-a)$.

Any abelian group is a tropical group.

Theorem. $(\mathbb{C}, \top)$ is a tropical group.

Oleg Viro, On basic notions of the tropical geometry, 2010.

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$$\{(k_1, \ldots, k_t) \to (k_2, \ldots, k_{t+1}) : \; k_{t+1} \in f(k_1, \ldots, k_t)\}.$$

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$$\mathrm{M}_f(A_1, \ldots, A_t) = (A_2, \ldots, A_t, \cup_{k_1 \in A_1, \ldots, k_t \in A_t} f(k_1, \ldots, k_t)).$$

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The phase space of f, denoted by $\mathcal{PS}_f$, has vertex set $(2^K)^t$ and arc set $\{A \to \mathrm{M}_f(A) : \ A \in (2^K)^t\}$. The structure of $\mathcal{PS}_f$ tells us the dynamical behavior of M_f .

Fusion rule of Majorana algebra

	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	a,b	d
d	d	d	d	a,b,c

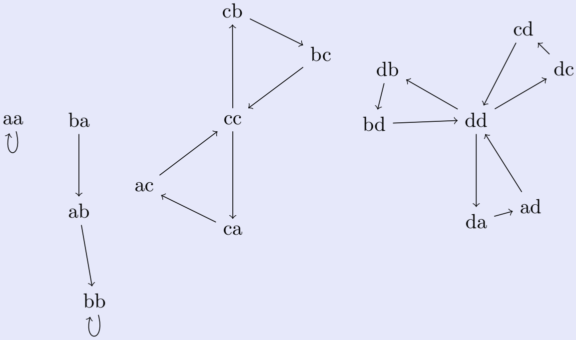

Figure 2: De Bruijn form

例

Let $A_1, \ldots, A_n$ be the adjacency matrices of the binary relations in a coherent configuration. Let $K = \{A_1, \ldots, A_n\}$. For any $(A_i, A_j) \in K^2$, let $f(A_i, A_j) = \{A_k :$ the intersection number p_{ij}^k is nonzero$\}$. Then f is a 2-variables Boolean digraph.

Similarly, we can often obtain from any t-ary operation a t-variables Boolean digraph.

A coherent algebra and the corresponding 2-variables digraph

$$\begin{bmatrix} D & E & F \\ G & H & G \\ F & E & D \end{bmatrix}$$

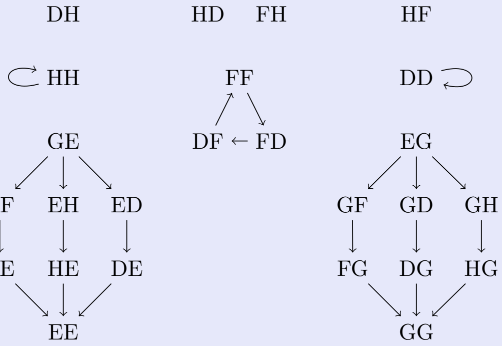

Figure 3: De Bruijn form

$$f(11) = \{2\}, f(12) = \{1, 2\}, f(21) = \{1\}, f(22) = \emptyset$$

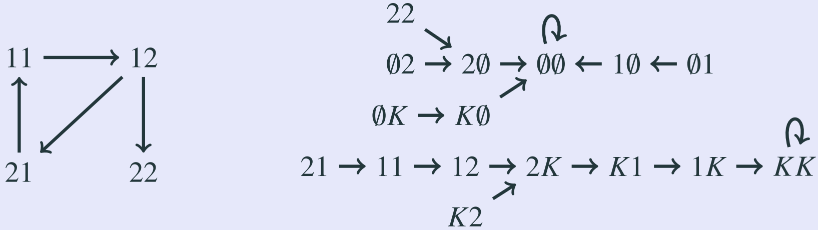

Figure 4: The De Bruijn form and the phase space of a 2-variables digraph f on $K = \{1, 2\}$.

We call $A \in (2^K)^t$ transient for f if it is not on any limit cycle of $\mathcal{PS}_f$.

Qubit entanglement

$(2^K)^t$ is exactly the set of those untangled elements in $2^{(K^t)}$, or the set of t-boxes in $2^{(K^t)}$. Only when $t = 1$, we have $(2^K)^t = 2^{(K^t)}$!

How to tell the phase transitions at the level of untangled elements from the local picture of the De Bruijn form?

Wave-particle duality

One can view Γ_f as a local dynamical mechanism and $\mathcal{PS}_f$ as the global evolving picture. The question is to see how to link the local with the global.

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One can also think of Γ_f as the particle version of f and $\mathcal{PS}_f$ as the wave version of f. Both versions encode full information about f in some way.

Figure 5: Particle or wave?

An expansion property

Figure 6: Helmut Wielandt (1910–2001)

This research area seems to me to be a mountainous region that is still undisturbed by roads and has to be traversed on foot. But this has its charm. And the nice surprises that one experiences compensate for the occasional compassionate glances of motorists. – Helmut Wielandt, 1961/62.

A digraph f on a set K is primitive whenever there exists an integer N such that $\mathrm{M}_f^N(a) = K$ for all $a \in K$. The smallest such N for a primitive digraph f is called the primitive exponent of f.

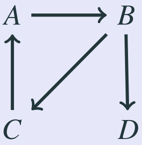
If f has primitive exponent γ, then every walk in $\mathcal{PS}_f$ of length no shorter than γ and starting at any vertex other than $\emptyset$ will end at K.

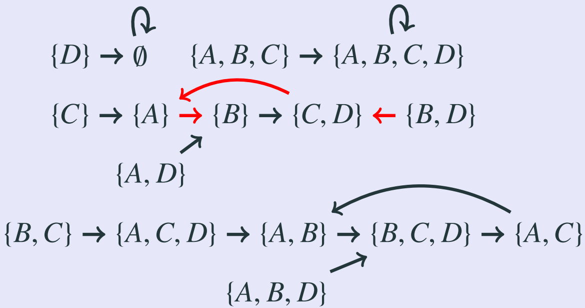
The Wielandt digraph W_m on m vertices is an m-cycle plus an additional arc that makes an $(m-1)$-cycle.

定理 (Wielandt, 1959 (1950?); Schneider, 2002)
Let K be a finite set of size m. A digraph f on K is primitive if and only if $\mathrm{M}_f^{(m-1)^2+1}(a) = K$ for all $a \in K$. A primitive digraph on K has primitive exponent $(m-1)^2 + 1$ if and only if it is the Wielandt digraph W_m.

Wielandt's best known paper in matrix theory revitalized the area of nonnegative matrices and generalizations. It appeared in November 1950 which, as it happens, is also the month when I started research in this area. – Hans Schneider

Figure 7: Hans Schneider (1927–2014)





$\mathrm{M}_f^3(\{B, D\}) = \{B\}$

Figure 8: A digraph f on $K = \{A, B, C, D\}$ and its phase space $\mathcal{PS}_f$.

定理 (Coxson-Larson-Schneider, 1987)
Let $f : K \to 2^K$ be a digraph. If $Y = \mathrm{M}_f^m(X)$ is a singleton set, then there exists $h \leq |K| - 1$ such that $Y = \mathrm{M}_f^h(X)$.

Reaching a general set

Let f be a digraph on K. For any $u, v \in K$, let

$$\mathrm{Dist}_f(u, v) = \min\{h \geq 0 : \ v \in \mathrm{M}_f^h(u)\}.$$

Let

$$\mathrm{Dia}_f = \max_{u,v \in K} \mathrm{Dist}_f(u, v).$$

We call f strongly connected if $\mathrm{Dist}_f(u, v) < \infty$ for all $u, v \in K$. If f is strongly connected, then $\mathrm{Dia}_f \leq |K| - 1$.

定理 (W.-Xu-Zhu, 2016)
Let f be a strongly connected digraph. If $Y = \mathrm{M}_f^h(X)$ is transient, then $h \leq |Y| \, \mathrm{Dia}_f$.

Poset structure on t-boxes

For $A = A_1 \times \cdots \times A_t, B = B_1 \times \cdots \times B_t \in (2^K)^t$, we write $A \leq B$ whenever $A_i \subseteq B_i$ for $i = 1, \ldots, t$.

This makes the set of t-boxes on K (namely $(2^K)^t$) a poset with the maximum element

$$(\underbrace{K, \ldots, K}_{t}) = K^t$$

and the minimum element $\emptyset^t$. An element from $(2^K)^t$ which has $\emptyset$ as one of its t components is called a vacant element.

Warning: Distinguish the usage of K^t as a point and as a set $\binom{K}{1}^t$.

Distance function for multivariate digraphs

Let f be a t-variables digraph on K.

For any $u, v \in K^t = \binom{K}{1}^t$, let

$$\mathrm{Dist}_f(u, v) = \min\{h \geq 0 : \ v \leq \mathrm{M}_f^h(u)\}.$$

We call f strongly connected if $\mathrm{Dist}_f(u, v) < \infty$ for all $u, v \in K^t$.

The diameter of f, denoted by Dia_f, is $max_{x \neq y} \mathrm{Dist}_f(x, y)$. Let $D_{t,k} := \max \left\{ \mathrm{Dia}_f \mid f \text{ is a strongly connected } t\text{-digraph on } [k] \right\}$.

定理 (W.-Xu-Zhu, 2017)
For $k \geq 5$,

$$D_{2,k} \geq \begin{cases} 2k^2, & \text{if } k \text{ is odd;} \\ 2k^2 - k + 1, & \text{if } k \text{ is even.} \end{cases}$$

For $A = A_1 \times \cdots \times A_t \in (2^K)^t$, let the length of A be $\|A\| = \sum_{i=1}^t |A_i|$.
For example, $\|X\| = t$ for each $X \in K^t$.

定理 (Chen-W., 2018+; generalizing W.-Xu-Zhu2016)
Let f be a t-variables strongly connected digraph. Let $X \in K^t = \binom{K}{1}^t$ be an element in a sink component of Γ_f and let D be the diameter of that component. If $Y = \mathrm{M}_f^h(X)$ is transient for f, then $h \le (\|Y\| - t + 1)D$.

定理 (Chen-W., 2018+)
Let f be a t-variables strongly connected digraph. Take $X \in K^t$. If $Y = \mathrm{M}_f^h(X)$ is transient for f, then $h \le (\|Y\| - t + 1)(k^t - 1)$.

Primitive digraphs

Let f be a t-variables digraph on K. It is clear that every vacant element of $(2^K)^t$ will reach $\emptyset^t$ in $\mathcal{PS}_f$ in $t-1$ steps.

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We call f primitive provided every element from $\binom{K}{1}^t \subseteq (2^K)^t$ will reach $K^t \in (2^K)^t$ in $\mathcal{PS}_f$, equivalently, every non-vacant element will go to the largest element K^t in $\mathcal{PS}_f$ and every vacant element will go to the minimum one $\emptyset^t$ in $\mathcal{PS}_f$.

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For a primitive t-variables digraph f on K, we define

$$\gamma(f) = \min\{i > 0 : \mathrm{M}_f^i(A) = K^t \text{ for all } A \in K^t\}$$

and call it the primitive exponent of f.

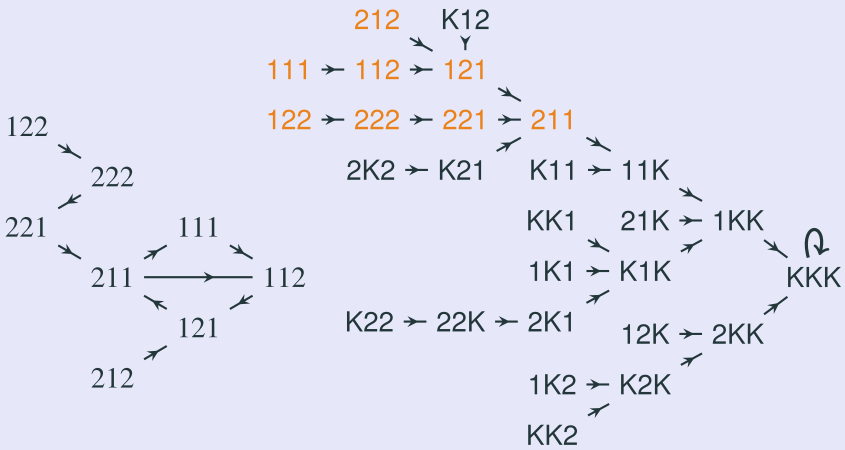

Figure 9: The De Bruijn form and part of the phase space induced by non-vacant vertices of a primitive 3-variables digraph on $K = [2]$. This digraph is not primitive in the sense of Chang-Pearson-Zhang (SIMAX, 2011).

Maximum primitive exponent

Let $\mathscr{D}_{t,k}$ be the set of primitive t-variables digraphs on $[k]$ and let

$$\gamma(t,k) := \max_{f \in \mathscr{D}_{t,k}} \gamma(f).$$

- Wielandt's bound (1959?): $\gamma(1,k) = (k-1)^2 + 1$.
- $\gamma(2,1) = 1, \gamma(2,2) = 7, \gamma(2,3) = 23 = (2 \times 3 - 1)^2 - 2$.
- Is it true that $\gamma(2,k) = (2k-1)^2 + 1$ when $k \geq 4$?
- Let r_k be the minimum number of multiplications to multiply two k by k matrices (rank of the k by k matrix multiplication tensor). By coincidence (?), $r_1 = 1, r_2 = 7, r_3 \leq 23$. Can we expect $r_k \leq \gamma(2,k)$?

定理 (W.-Xu-Zhu, 2017)

- $(2k-1)^2 + 1 \le \gamma(2,k)$ *when* $k \ge 4$.
- $k^t \le \gamma(t,k)$ *for all positive integers* k *and* t.

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- $(2k-1)^2 + 1 \leq \gamma(2,k)$ *when* $k \geq 4$.
- $k^t \leq \gamma(t,k)$ *for all positive integers* k *and* t.

Conjecture (W.-Xu-Zhu, 2016)
$\gamma(2,k) = O(k^2)$.

定理 (Chen-W., 2018+)
For all positive integers t and k,

$$\gamma(t, k) \leq t(k-1)(k^t - 1) + 1.$$

When $t = 1$, this upper bound is just Wielandt's bound.

定理 (Chen-W., 2018+)
For all positive integers t and k,

$$\gamma(t,k) \le t(k-1)(k^t-1)+1.$$

When $t=1$, this upper bound is just Wielandt's bound.

When $(t,k)=(2,2)$, the bound coincides with

$$\gamma(2,2) = 2\times(2-1)\times(2^2-1)+1 = 7.$$

$\gamma(2,2) = 2 \times (2-1) \times (2^2 - 1) + 1 = 7$

A 2-variables digraph f on $K = \{1, 2\}$:

$$\begin{cases} f(1,1) = & \{2\}; \\ f(1,2) = & \{2\}; \\ f(2,1) = & \{1,2\}; \\ f(2,2) = & \{1\}. \end{cases}$$

$$\begin{tikzcd}
11 \arrow[r] & 12 \arrow[d] \\
21 \arrow[u] \arrow[ur] & 22 \arrow[l]
\end{tikzcd}$$

Figure 10: De Bruijn form Γ_f

$$\begin{tikzcd}
11 \arrow[r] & 12 \arrow[r] & 22 \arrow[r] & 21 \arrow[d] \\
 & & K1 \arrow[r] & 1K \arrow[d] \\
 & KK \arrow[loop above] & 2K \arrow[l] & K2 \arrow[l]
\end{tikzcd}$$

Figure 11: Part of phase space $\mathcal{PS}_f$

Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a set of n digraphs on K. The phase space of $\mathcal{F}$, denoted by $\mathcal{PS}_{\mathcal{F}}$, has 2^K as vertex set and has its arc set to be the union of the arc sets of $\mathcal{PS}_{f_i}, i = 1, \ldots, n$.

Conjecture (W.-Xu-Zhu, 2016)
If $A \in 2^K$ can reach $B \in 2^K$ in $\mathcal{PS}_{\mathcal{F}}$, then A can reach B in $\mathcal{PS}_{\mathcal{F}}$ in at most $|B||K|^n$ steps.

We have no clue how to tackle this conjecture and so we avoid discussing any of its counterparts for digraphs of several variables.

Thank you for listening!

Figure 12: Novosibirsk, G2R2, 2018

Figure 13: Yichang, G2D2, 2019