Weakly Distance-Regular Digraphs

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Outline

Digraph

- A digraph Γ is a pair (X, A) where X is a finite set of vertices and $A \subseteq X \times X$ is a set of arcs.

Path and Distance

- A path from u to v:

$$(u = w_0, w_1, \ldots, w_r = v)$$

  such that (w_{t-1}, w_t) is an arc for $t = 1, 2, \ldots, r$.
- r is the length of the path (the number of arcs).
- Distance $\partial(u, v)$: the length of a shortest path from u to v.
- The maximum value of distance function is the diameter of Γ.

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Circuit

- The path $(w_0, w_1, \dots, w_{r-1})$ is a circuit if (w_{r-1}, w_0) is an arc.
- The girth of Γ is the length of a shortest circuit.

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- The girth of Γ is the length of a shortest circuit.

Strongly connected

- A digraph Γ is said to be strongly connected if, for any two vertices x and y, there is a path from x to y.

Cayley digraph

- G: a finite group; $1 \notin S \subseteq G$.
- The Cayley digraph $\mathrm{Cay}(G, S)$:
  the vertex set is G,
  (x, y) is an arc if, there exists an $s \in S$ such that $y = sx$.
- A digraph which is isomorphic to a Cayley digraph of a cyclic group is called a circulant digraph.

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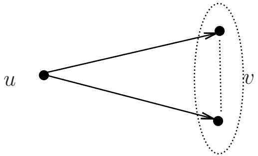
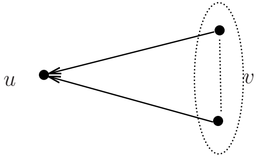
Regularity

- Outdegree of u, denoted by $d^+(u)$:



- Indegree of u, denoted by $d^-(u)$:



- Γ is k-regular, if $d^+(u) = d^-(u) = k$ for any u.

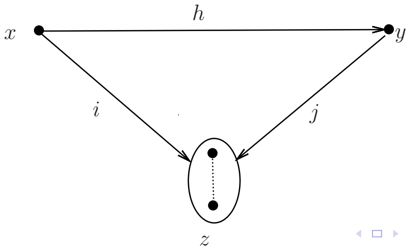
DRG; Famous, well studied

Definition (1970's)

A connected graph Γ is said to be distance-regular, if for any vertices x, y with $\partial(x, y) = h$, the number

$$p^h_{i,j} = |\{z \in V\Gamma \mid \partial(x, z) = i, \partial(y, z) = j\}|$$

depends only on i, j, h.



Directed version

How to define the directed version?

DRDG

Definition (Damerell, 1981 [1])

A strongly connected digraph Γ is said to be distance-regular, if for any vertices x, y with $\partial(x,y) = h$, the number

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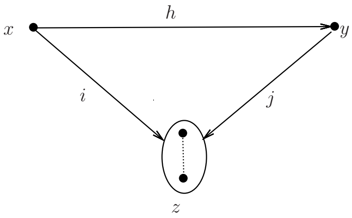

R.M. Damerell, Distance-transitive and distance regular digraphs, J. Combin. Theory Ser. B, 31 (1981) 46–53.

Construction of DRDG

It is difficult to construct a new DRDG! For almost all constructions, see:

- H, Enomoto, R.A. Mena, Distance-regular digraphs of girth 4, J. Combin. Theory Ser. B 43 (1987), 293–302.
- R.A. Liebler, R.A. Mena, Certain distance-regular digraphs and related rings of characteristic 4, J. Combin. Theory Ser. A 47 (1988), 111–123.
- M. Yamada, Distance-regular digraphs of girth 4 over an extension ring of $Z/4Z$, Graphs Combin. 6 (1990), 381–394.
- T. Takahashi, Distance-regular digraphs of girth 6, Mem. Fac. Sci. Kyushu Univ. Ser. A 45 (1991), 155–166.

Rarity of DRDG

- In 1981, E. Bannai, P.J. Cameron, and J. Kahn proved that a distance-transitive digraph (a little stronger than DRDG) of odd girth is a Paley tournament, a directed cycle, or its coclique extension.
- In 1991, A. Munemasa studied nonsymmetric P-and Q-polynomial association schemes, proved that a DRDG with Q-polynomial property is a directed cycle of length five if its girth is five.

• E. Bannai, P.J. Cameron and J. Kahn, Nonexistence of certain distance-transitive digraphs, J. Combin. Theory Ser. B, 31 (1981), 105–110.

• A. Munemasa, On nonsymmetric P-and Q-polynomial association schemes, J. Combin. Theory Ser. B 51 (1991), 314–328.

Rarity of DRDG (cont.)

- In 1993, D.A. Leonard and K. Nomura proved that except directed cycles and their coclique extension all DRDG have $d = g - 1 \leq 7$.
- The above results imply that DRDG are very few. Since then, The study for DRDG has been stopped.

• D.A. Leonard and K. Nomura, The girth of a distance-regular digraph, J. Combin. Theory Ser. B 58 (1993), 34–39.

Reason

In a distance-regular digraph,

$$\partial(x,y)=\partial(x',y')\Longrightarrow\partial(y,x)=\partial(y',x').$$

BUT it does not hold in general.

Two way distance

- Two way distance $\tilde{\partial}(u,v) = (\partial(u,v), \partial(v,u))$.
- $\tilde{\partial}(\Gamma) = \{\tilde{\partial}(x,y) \mid x,y \in V\Gamma\}$.
- An arc (u,v) is of type $(1,r)$ if $\tilde{\partial}(u,v) = (1,r)$.

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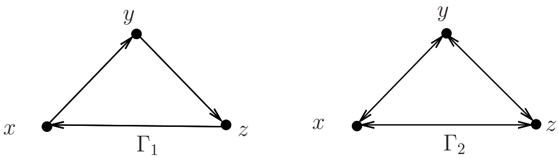
Examples



Examples:

- In Γ_1, $\tilde{\partial}(x, y) = (1, 2)$, (x, y) is an arc of type $(1, 2)$.
- In Γ_2, $\tilde{\partial}(x, y) = (1, 1)$, (x, y) is an arc of type $(1, 1)$.

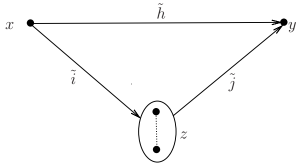
Wdrdg

Definition (Wang and Suzuki, 2003)

A strongly connected digraph Γ is said to be weakly distance-regular (wdrdg) if, for any $\tilde{\partial}(x,y)=\tilde{h}$,

$$p^{\tilde{h}}_{\tilde{i},\tilde{j}}=|\{z\in V\Gamma\mid \tilde{\partial}(x,z)=\tilde{i}\text{ and }\tilde{\partial}(z,y)=\tilde{j}\}|$$

depends only on $\tilde{i},\tilde{j},\tilde{h}$.



K. Wang and H. Suzuki, Weakly distance-regular digraphs, Discrete Math. 264 (2003) 225-236.

The attached scheme

- Let Γ be a wdrdg.
- Let $\Gamma_{\tilde{i}} = \{(x,y) \in V\Gamma \times V\Gamma \mid \tilde{\partial}(x,y) = \tilde{i}\}$. Then $(V\Gamma, \{\Gamma_{\tilde{i}}\}_{\tilde{i} \in \tilde{\partial}(\Gamma)})$ is an association scheme. The scheme is called the attached scheme of Γ.
- $p^{\tilde{h}}_{\tilde{i},\tilde{j}}$: intersection number.
- Γ is regular.

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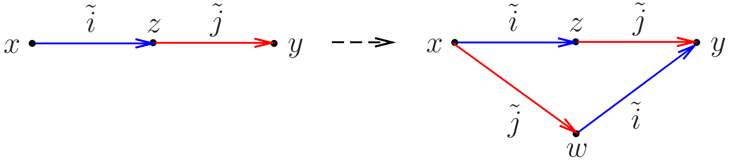
Commutativity

- Γ is commutative if the attached scheme is commutative, that is

$$p^{\tilde{h}}_{\tilde{i},\tilde{j}} = p^{\tilde{h}}_{\tilde{j},\tilde{i}}, \ \forall\, \tilde{i}, \tilde{j}, \tilde{h}.$$

It means



Outline

Valency two

Theorem (Wang and Suzuki 2003)

A commutative wdrdg of valency 2 is isomorphic to one of the following Cayley digraphs:

(1) $\mathrm{Cay}(\mathbb{Z}_3^2, \{(0,1),(1,0)\})$.

(2) $\mathrm{Cay}(\mathbb{Z}_{2n}, \{1,2\})$.

(3) $\mathrm{Cay}(\mathbb{Z}_{2n}, \{1,n+1\})$.

(4) $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_n, \{(0,1),(1,0)\})$.

In 2004, Suzuki proved the nonexistence of noncommutative 2-valent wdrdgs.

K. Wang and H. Suzuki, Weakly distance-regular digraphs, Discrete Math. 264 (2003) 225-236.

H. Suzuki, Thin weakly distance-regular digraphs, J. Combin. Theory Ser. B 92 (2004), 69-83.

Thin

- A wdrdg is thin if the maximum value of its intersection numbers is 1.
- In 2004, Suzuki classified all **thin** wdrdgs.

Theorem (Suzuki 2004)

A thin wdrdg is isomorphic to one of the following Cayley digraphs:

(1) $\mathrm{Cay}(\mathbb{Z}_n, \{1\})$.

(2) $\mathrm{Cay}(\mathbb{Z}_{2n}, \{1, 2\})$.

(3) $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_n, \{(1, 0), (0, 1)\})$.

(4) $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2n}, \{(1, 0), (0, 1), (0, 2)\})$.

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Problem by Suzuki

- It seems very interesting to classify wdrdgs of valency three.

H. Suzuki, Thin weakly distance-regular digraphs, J. Combin. Theory Ser. B 92 (2004), 69-83.

Cubic, girth 2

Theorem (Wang 2004)

A commutative wdrdg of valency 3 and girth 2 is isomorphic to one of the following digraphs:

(1) $\mathrm{Cay}(\mathbb{Z}_{2n}, \{1, n, n+1\})$.

(2) $\mathrm{Cay}(\mathbb{Z}_{4n}, \{1, 2n-1, 2n\})$.

(3) $\mathrm{Cay}(\mathbb{Z}_{2n} \times \mathbb{Z}_2, \{(0,1), (1,0), (2,0)\})$.

(4) $\mathrm{Cay}(\mathbb{Z}_{2n} \times \mathbb{Z}_2, \{(0,1), (1,0), (n+1,0)\})$.

(5) $\mathrm{Cay}(\mathbb{Z}_{4n} \times \mathbb{Z}_2, \{(0,1), (1,0), (4n-1,1)\})$.

(6) $\mathrm{Cay}(\mathbb{Z}_{n} \times \mathbb{Z}_m, \{(0,1), (1,0), (n-1,0)\})$.

(7) $\mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2, \{(1,0,0), (0,1,0), (0,0,1)\})$.

K. Wang, Commutative weakly distance-regular digraphs of girth 2, Europ. J. Combin., 25(2004), 363-375.

Outline

Classification scheme

Γ: wdrdg of valency three and girth $g > 2$.
Then Γ has at most three types of arcs.
In order to classify wdrdgs of valency three, we divide our discussion into three cases:

- Three types of arcs. (Γ is thin)
- Two types of arcs.
- One type of arcs.

Two types of arcs

Theorem, (Y., Lv, Wang, 2016)

Let Γ be a wdrdg of valency 3 and girth more than 2. If Γ has two types of arcs, then Γ is isomorphic to one of the following digraphs:

(1) $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_g, \{(0,1),(2,1),(1,0)\})$, where $g = 3$ or $g \geq 5$.

(2) $\mathrm{Cay}(\mathbb{Z}_q \times \mathbb{Z}_{2mq}, \{(1,0),(0,1),(1,2mq-1)\})$, $m \geq 1$ and $q \geq 3$.

(3) $\mathrm{Cay}(\mathbb{Z}_{(mq+2)q}, \{1, mq+2, mq+1\})$, $m \geq 1$ and $q \geq 3$.

(4) $\mathrm{Cay}(\mathbb{Z}_{2^i q} \times \mathbb{Z}_{2^{-i}(2mq+p)}, \{(2^i, ih), (2^i ud, 1), (2^i - 2^i ud, ih-1)\})$, where $q \geq 3$, $m \geq 0$, $4 \leq p \leq 2q-2$ and p is even.

Let $d = \frac{p}{2(q,p)}$, $l = \max\{w \mid 2^w \text{ divides } (q,p)\}$, $h = \frac{s}{2^l}$, $i = 2\{d\}$ and u be an integer such that $2^i q$ divides $(up - (q,p))$, where $\{d\}$ denotes the fractional part of d and (q,p) denotes the greatest common divisor of q and p.

Y. Yang, B. Lv and K. Wang, Weakly distance-regular digraphs of valency three I, Electron. J. Combin. 23(2) (2016), Paper 2.12.

One type of arcs

Theorem, (Y., Lv, Wang, 2018)

Let Γ be a commutative wdrdg of valency 3 and girth more than 2. If Γ has only one type of arcs, then Γ is isomorphic to:

(1) $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$.
(2) $\mathrm{Cay}(Q_8, \{i, j, k\})$, Q_8 is the quaternion group of order 8.
(3) $\mathrm{Cay}(\mathbb{Z}_{13}, \{1, 3, 9\})$.
(4) The eighteenth digraph with 18 vertices in [1].
(5) $\mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\})$.
(6) $\mathrm{Cay}(\mathbb{Z}_g \times \mathbb{Z}_3, \{(1, 0), (1, 1), (1, 2)\})$.
(7) $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_n, \{(1, 0), (0, 1), (n-1, n-1)\})$, $n \notin 3\mathbb{Z} \setminus \{3\}$.
(8) $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_{3n}, \{(0, 1), (1, 1), (n-1, 3n-2)\})$, $n \geq 2$.

A. Hanaki, Classification of weakly distance-regular digraphs with up to 21 vertices, http://math.shinshu-u.ac.jp/∼hanaki/as/data/wdrdg.

Y. Yang, B. Lv and K. Wang, Weakly distance-regular digraphs of valency three, II, J. Combin. Theory Ser. A 160 (2018) 288–315.

Is commutativity redundant?

- All commutative cubic wdrdgs are classified!
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Quasi-thin

- A wdrdg is quasi-thin if the maximum value of its intersection numbers is 2.

Theorem, (Y., Lv, Wang, 2018+)

If Γ is a commutative quasi-thin wdrdg of valency more than 3, then Γ is isomorphic to one of the following Cayley digraphs:

(i) $\mathrm{Cay}(\mathbb{Z}_8, \{1, 2, 3, 6\})$.

(ii) $\mathrm{Cay}(\mathbb{Z}_{4p}, \{1, 2, 2p + i, 2p + 1, 2p + 2\})$, $p \neq 2 - i$.

(iii) $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_4, \{(0, 1), (1, 0), (2, 0), (0, 2)\})$.

(iv) $\mathrm{Cay}(\mathbb{Z}_q \times \mathbb{Z}_4, \{(0, 1), (1, 0), (1, 2), (0, 2 + i)\})$, $q \neq 3 + i$.

(v) $\mathrm{Cay}(\mathbb{Z}_{2q} \times \mathbb{Z}_2, \{(0, 1), (1, 0), (2, 0), (1, 1)\})$.

Quasi-thin

(vi) $\mathrm{Cay}(\mathbb{Z}_{4q}\times\mathbb{Z}_2,\{(0,1),(1,0),(2,0),(2q+1,0),(2q+2,0),(2qi,1)\})$, $q\notin\{3,3+i\}$.

(vii) $\mathrm{Cay}(\mathbb{Z}_{2q}\times\mathbb{Z}_4,\{(0,1),(1,0),(1,2),(0,2-i),(2,0),(2,2)\})$, $q\notin\{3,3+i\}$.

(viii) $\mathrm{Cay}(\mathbb{Z}_{2q}\times\mathbb{Z}_n,\{(0,1),(1,0),(2,0),(0,-1)\})$.

(ix) $\mathrm{Cay}(\mathbb{Z}_{2q}\times\mathbb{Z}_n,\{(0,1),(1,(c+1)/2),(1,(c-1)/2),(2,c),(0,-1)\})$.

(x) $\mathrm{Cay}(\mathbb{Z}_{2n}\times\mathbb{Z}_q,\{(0,1),(1,(t+1)/2),(-1,(1-t)/2),(2,t),(-2,-t)\})$.

Here, $i\in\{0,1\}$, $2\leq p$, $3\leq q$, $3\leq n\leq q-(1+(-1)^q)/2$, $c=n/\gcd(q,n)$, $t=q/\gcd(q,n)$ and c,t are both odd.

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Y. Yang, B. Lv and K. Wang, Quasi-thin weakly distance-regular digraphs, arXiv:1609.04962.

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Primitive weakly distance-regular circulant digraphs

- A wdrdg Γ is primitive if the digraph $(V\Gamma, \Gamma_{\tilde{i}})$ is strongly connected for any $\tilde{i} \in \tilde{\partial}(\Gamma) \backslash \{(0,0)\}$.

Theorem, (Munemasa, Wang, Y. 2018+)

If Γ is a primitve weakly distance-regular circulant digraph, then Γ is isomorphic to one of the following digraphs:

(i) the Paley digraph of order p, where p is prime such that $p \equiv 3 \pmod 4$.

(ii) the circuit of length p, where p is prime.

(iii) $\mathrm{Cay}(\mathbb{Z}_{13}, \{1,3,9\})$.

A. Munemasa, K. Wang and Y. Yang, A family of pseudocyclic association schemes, in preperation.

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Problems

Now I list some problems we are interested in.

Circulant digraph

Continued to characterize weakly distance-regular circulant digraphs.

Attached scheme is special

Study such a wdrdg whose attached association scheme is special, such as primitive, regular, P-polynomial, Q-polynomial.

Commutativity

- We do not know any example of noncommutative wdrdgs. Are there such digraphs?

Relation between wdrdg and DRG

- Study such a wdrdg whose base graph is a distance-regular graph.
- Study such a distance-regular graph which is the base graph of a wdrdg.

Hanaki's list

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Thank you for your attention.