Tropical hyperplane arrangement and zonotopal tiling

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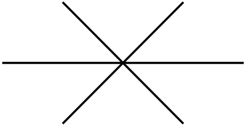
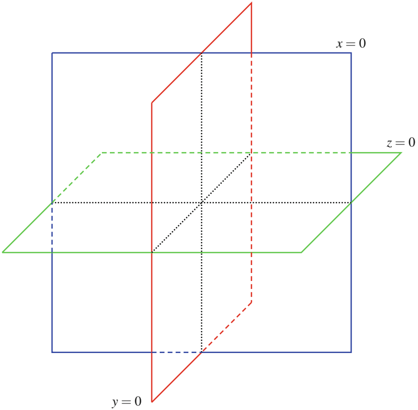
Hyperplane arrangement

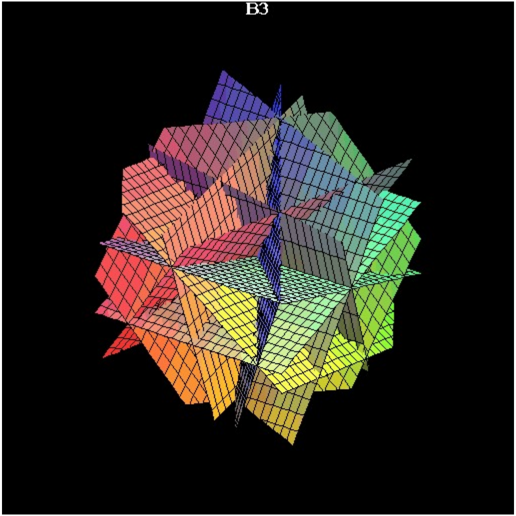
The zero set of a linear function

$$c_1 \cdot x_1 + \cdots + c_n \cdot x_n$$

is a *hyperplane* in $\mathbb{R}^n$. A *hyperplane arrangement* is a set of hyperplanes.







A hyperplane arrangement induced by roots of type B_3. The picture is from the homepage of John Stembridge.

Tropical hyperplane

Tropical algebra $(\mathbb{R}, \oplus, \odot)$: $a \oplus b := \max\{a, b\}$; $c \odot d := c + d$.

Tropicalization: $(+, \cdot) \Rightarrow (\oplus, \odot)$.

Tropical linear form

$$c_1 \cdot x_1 + \cdots + c_n \cdot x_n \Rightarrow c_1 \odot x_1 \oplus \ldots \oplus c_n \odot x_n \quad (\max\{c_1 + x_1, \ldots, c_n + x_n\})$$

A *tropical hyperplane* is the locus where a tropical linear form fails to be linear.

For each vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we define a tropical hyperplane $H_{\max}(v)$ by

$$\{x : \text{the maximum in } \{x_1 - v_1, \ldots, x_n - v_n\} \text{ is achieved at least twice}\}.$$

$$H_{\max}(v) = H_{\max}(0) + v$$

Tropical hyperplane

The tropical projective space $\mathbb{TP}^{n-1}$ is the quotient space $\mathbb{R}^n \,/\, \mathbb{R}\,\mathbf{1}_n$.

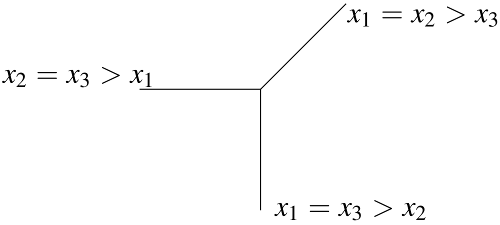
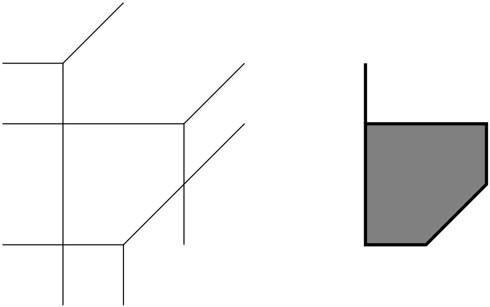


Figure: $H_{\max}(0)$ in $\mathbb{TP}^2$ represented by the plane $\{x \in \mathbb{R}^3 : x_3 = 0\}$.

Let $\mathcal{F}_n$ be the outer normal fan of the standard simplex $\Delta_n := \mathrm{conv}\{\delta_1, \ldots, \delta_n\}$ in $\mathbb{R}^n$. Then $H_{\max}(0)$ is the union of codimension-one cells of $\mathcal{F}_n$. We regard $\mathcal{F}_n$ as the subdivision of $\mathbb{R}^n$ induced by $H_{\max}(0)$.

Tropical complex

For a finite set V in $\mathbb{R}^n$, $\mathcal{H}_{\max}(V)$ denotes the tropical hyperplane arrangement consisting of $H_{\max}(v)$ with $v \in V$. The *tropical complex $\mathcal{C}_V$ generated by V* is the complex of bounded cells in the subdivision of $\mathbb{TP}^{n-1}$ induced by $\mathcal{H}_{\max}(V)$.



$\mathcal{H}_{\max}(V)$ $\mathcal{C}_V$

Figure: A tropical hyperplane arrangement and a tropical complex in $\mathbb{TP}^2$.

Proposition (Develin and Sturmfels,2004)

Every cell in a tropical complex is a Lipschitz polytope.

Lipschitz polytope

For each $n \times n$ square matrix D we define a polytope L_D by

$$L_D = \{x \in \mathbb{R}^n : x_i - x_j \leq D(i,j), 1 \leq i,j \leq n\}.$$

Polytopes in this form are called *Lipschitz polytopes*

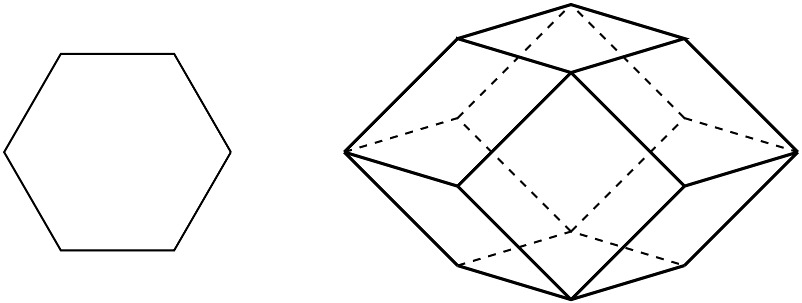


Figure: Lipschitz polytope of 3-point or 4-point metric D with $D(i,j) = 1$.

Here are some other names of Lipschitz polytopes:

- (dual of) fundamental polytope (Vershik)
- Polytrope (Kulas and Joswig)
- Alcoved polytope of type A_n (Lam and Postnikov)
- L-convex set (Murota)

AM Vershik "Classification of finite metric spaces and combinatorics of convex polytopes", Arnold Mathematical Journal, 2015.

Zonotope

A *zonotope* is the Minkowski sum of some line segments. Also, a zonotope is the image of a hypercube after an affine transformation.

Proposition

A polytope is a zonotope if and only if every of its dimension two face is centrally symmetric.

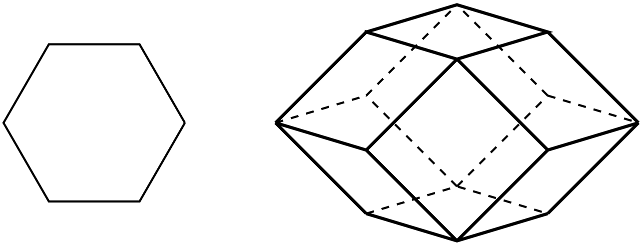
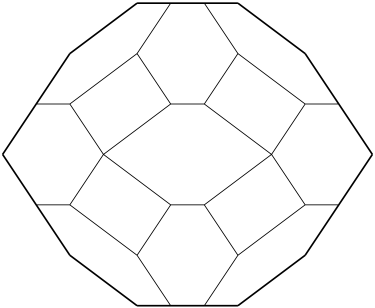


Figure: Lipschitz polytope of 3-point or 4-point metric D with $D(i,j) = 1$.

Zonotopal tiling

A zonotopal tiling is a subdivision of a zonotope by zonotopes.



Zonotopal Lipschitz polytope

Theorem (Godard, 2010; Wu and X., 2017)

Let P be a centrally symmetric polytope with center 0. Then P is a zonotope if and only if P equals L_D for some tree metric D.

Tree metric

A weighted X-tree $\mathcal{T} = (T, X, w, \ell)$ consistis of a tree T, a finite set X, a weight function $w : E(T) \to \mathbb{R}_{>0}$ and a labelling $\ell : X \to V(T)$.

$\mathcal{T}$ induces a metric $\mathrm{D}_{\mathcal{T}}$ on X :

$$\mathrm{D}_{\mathcal{T}}(x, y) = \sum_{e \in P_T(\ell(x), \ell(y))} w(e), \ \forall x, y \in X.$$

We call $\mathrm{D}_{\mathcal{T}}$ a tree metric, and we say $\mathcal{T}$ represents $\mathrm{D}_{\mathcal{T}}$.

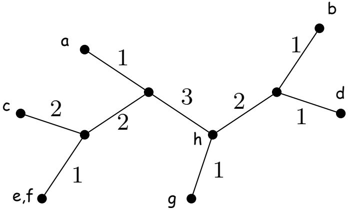


Figure: A weighted X-tree, $X = \{a, b, c, d, e, f, g, h\}$.

Tropical complex and zonotopal tiling

We say two $m \times n$ matrices D_1 and D_2 are *congruent* if there exists $f \in \mathbb{R}^m$ and $g \in \mathbb{R}^n$ such that

$$D_2(i,j) = D_1(i,j) + f_i + g_j.$$

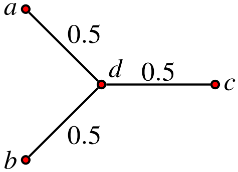
We say D' is a *row-submatrix* of a matrix D if rows of D' are rows of D.

$\mathcal{C}_D$ is the tropical complex generated by columns of D.

Theorem (Wu and X., 2018)

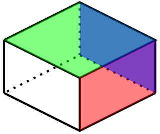
Let D be a matrix. Then $\mathcal{C}_D$ is a zonotopal tiling if and only if D is congruent with a row-submatrix of a tree metric.

Example



$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0.5 \\ 1 & 0 & 1 & 0.5 \\ 1 & 1 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{pmatrix}
\end{array}
$$

D



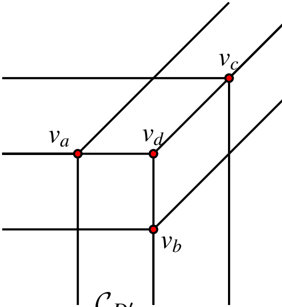
L_D

$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0.5 \\ 1 & 0 & 1 & 0.5 \\ 1 & 1 & 0 & 0.5 \end{pmatrix}
\end{array}
$$

D'

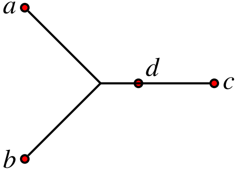
$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

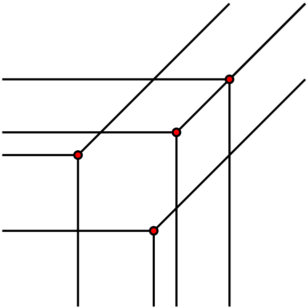
D''



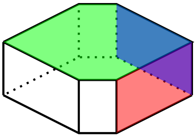
$\mathcal{C}_{D'}$

Example





$\mathcal{C}_{D'}$



L_D

A duality theorem

Theorem (Wu and X., 2018)

Let D' be a row-submatrix of a tree metric D. Then $\mathcal{C}_{D'}$ is piecewise linear isomorphic with a sub-complex of the face-complex of L_D.

- $\mathcal{C}_{D'}$ is piecewise linear isomorphic with $\mathcal{C}_{D'^\top}$ (Develin and Sturmfels, 2004);
- $\mathcal{C}_{D'^\top}$ is a sub-complex of the face-complex of L_D.

Combinatorics of zonotopal tiling

Let D' be a tree metric on Y represented by a weighted Y-tree (T, Y, w). Let $X \subseteq Y$ and let D be the row-submatrix of D' consisting of rows indexed by X.

A partial edge orientation σ of T is a *cyclic orientation with respect to* (T, X, Y) if:

1. every directed edge is on a directed path between two points from X in σ,
2. every point in $Y \setminus X$ can walk to a point in Y through directed edges in σ.

Denote by $\mathrm{CO}(T, X, Y)$ the poset on cyclic orientations of (T, X, Y) ordered under inclusion.

Theorem (Wu, X., 2018)

With above notations, the face poset of $\mathcal{C}_D$ is anti-isomorphic with $\mathrm{CO}(T, X, Y)$.

Remark: elements in $\mathrm{CO}(T, X, Y)$ correspond to covectors of the oriented matroid of the zonotopal tiling $\mathcal{C}_{\mathrm{D}}$.

Tropical convex set

A *tropical convex set* S in $\mathbb{R}^n$ is a set which satisfies: if $v, u \in S$ then $a \odot v \boxplus b \odot u \in S$ for any $a, b \in \mathbb{R}$. Here $\boxplus = \min$.

For a finite set V, the minimal tropical convex set $\operatorname{tconv}(V)$ which contains V is called the *tropical polytope generated by V*.

Theorem (Develin and Sturmfels, 2004)

For a finite set V, the tropical polytope $\operatorname{tconv}(V)$ is the support set of the tropical complex $\mathcal{C}_V$.

One-dimensional tropical polytope

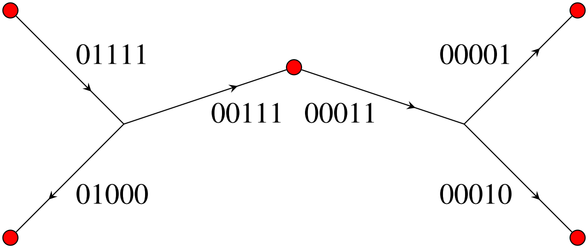


Figure: A 1-dimensional tropical polytope in $\mathbb{TP}^4$.

A one-dimensional tropical polytope is a tree such that

- the slope of every edge is a $0/1$-vector,
- the slopes of edges dicreases along each path in a same direction.

One-dimensional tropical linear space

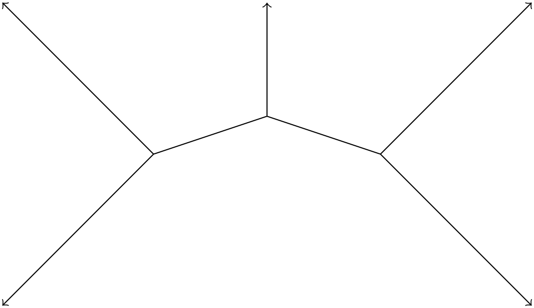


Figure: A one-dimensional tropical linear space in $\mathbb{TP}^4$.

A one-dimensional tropical linear space in $\mathbb{TP}^{n-1}$ is an infinite tree which is tropical convex and contains n rays.

Zonotopal filing

A zonotopal filing of $\mathbb{TP}^{n-1}$ is a tessellation of $\mathbb{TP}^{n-1}$ by zonotopes.

Theorem (Wu and X., 2018)

Let V be an infinite discrete set in $\mathbb{TP}^{n-1}$. Then $\mathcal{C}_V$ is a zonotopal filing if and only if $-V$ is on a one-dimensional tropical linear space L and each ray of L contains infinitely many points from $-V$.

Thank you for your attention