Phase spaces and kernel spaces of transformation semigroups

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August 18, 2018

Permutation groups

Let Ω be a finite set.

- The symmetric group on Ω, written $\mathrm{Sym}(\Omega)$, is the set of all permutations of Ω.
- A permutation group on Ω is a subgroup of $\mathrm{Sym}(\Omega)$.

Livingstone and Wagner theorem

Let Ω be a finite set. Let G be a permutation group on Ω.

- $\binom{\Omega}{k} := \{k\text{-element subsets of } \Omega\}$.
- G induces a permutation group G_k on $\binom{\Omega}{k}$.
- G is k-homogeneous if G_k is transitive.

Theorem (Livingstone-Wagner, 1965)

If $k \leq \ell$ and $k + \ell \leq |\Omega|$, then

$$\#\text{orbits of } G_k \leq \#\text{orbits of } G_\ell.$$

In particular,

$$\ell\text{-homogeneous} \Rightarrow k\text{-homogeneous}.$$

Transformation semigroup

Let Ω be a finite set.

- The full transformation monoid on Ω, written $T(\Omega)$, is the set of all maps from Ω to Ω.
- A transformation semigroup on Ω is a sub-semigroup of $T(\Omega)$.

Reachable digraphs

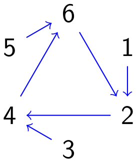
Let S be a transformation semigroup on Ω. Define the reachable digraph Γ_S of S to be the digraph with

- vertex set Ω;
- arc set $\{(x, s(x)) \mid x \in \Omega, s \in S\}$.

We call S transitive if Γ_S is strongly connected.

For a map $s : \Omega \to \Omega$, define

$$\begin{array}{rccc} \overline{s}: & 2^{\Omega} & \to & 2^{\Omega} \\ & X & \mapsto & \{s(x) \mid x \in X\}. \end{array}$$



s

$$\begin{tikzcd} \{1,2,3,4\} \arrow[r] & \{2,4,6\} \arrow[loop above] \end{tikzcd}$$

$\overline{s}$

Phase space, I

Let S be a transformation semigroup on Ω.

S induces a transformation semigroup $\overline{S} := \{\overline{s} \mid s \in S\}$ on 2^{Ω}. We call $\overline{S}$ the phase space of S.

We call S k-homogeneous if for all $X, Y \in \binom{\Omega}{k}$ there exists $s \in S$ such that $\overline{s}(X) = Y$.

Phase space, II

Let Γ be a digraph and $X \subseteq V_\Gamma$.

- $\Gamma[X] :=$ the induced subgraph of Γ on X.
- $\mathsf{wcc}(\Gamma) := \{$weakly connected compoent of $\Gamma\}$.

Theorem (Wu-Z.,2018+)

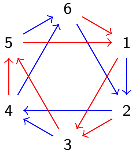
If $k \le \ell$ and $k + \ell \le |\Omega|$, then

$$\#\,\mathsf{wcc}(\Gamma_{\overline{S}}[\binom{\Omega}{k}]) \le \#\,\mathsf{wcc}(\Gamma_{\overline{S}}[\binom{\Omega}{\ell}]).$$

Moreover,

$$\ell\text{-homogeneous} \Rightarrow k\text{-homogeneous}.$$

An example of a phase space



$$\Omega = \{1, 2, \ldots, 6\}$$

$$S = \langle r, b \rangle$$

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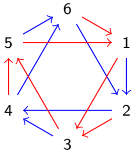

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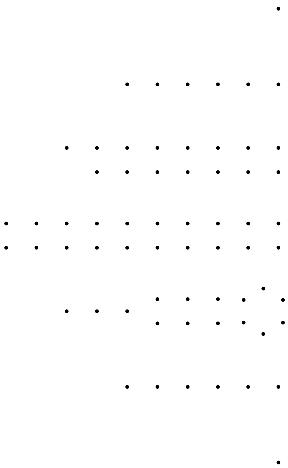

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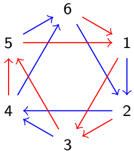
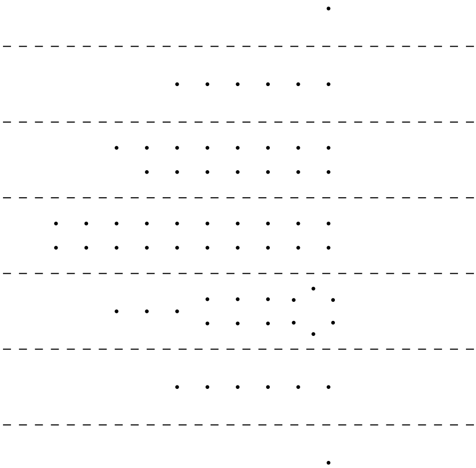


$$\Omega = \{1, 2, \ldots, 6\}$$

$$S = \langle {\color{red} r}, {\color{blue} b} \rangle$$



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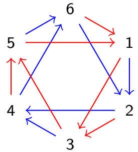

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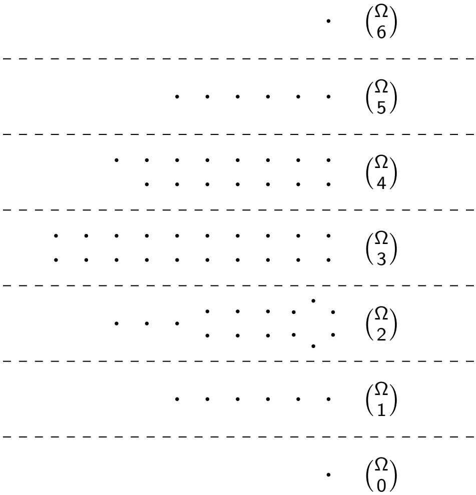
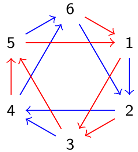
Vertex set



An example of a phase space



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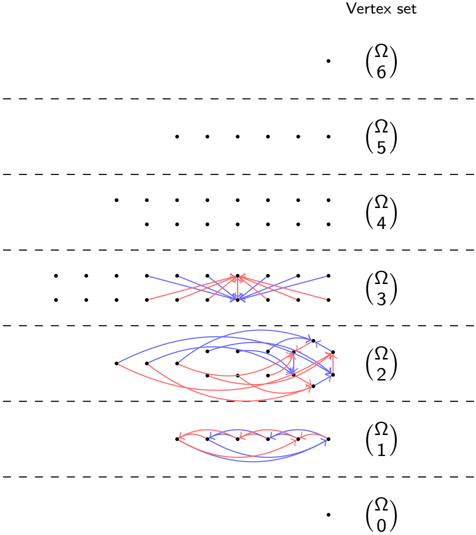

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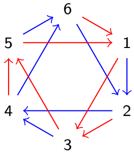

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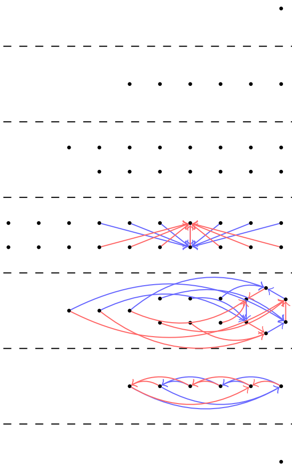
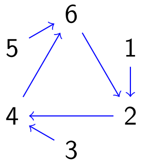


Vertex set	#wcc
$\binom{\Omega}{6}$	1
$\binom{\Omega}{5}$	6
$\binom{\Omega}{4}$	15
$\binom{\Omega}{3}$	7
$\binom{\Omega}{2}$	1
$\binom{\Omega}{1}$	1
$\binom{\Omega}{0}$	1

For a map $s : \Omega \to \Omega$, define

$$\begin{array}{rccc} s^{-1}: & 2^{\Omega} & \to & 2^{\Omega} \\ & X & \mapsto & \{y \mid s(y) \in X\}. \end{array}$$



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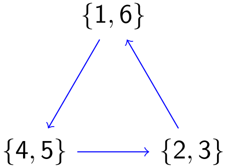
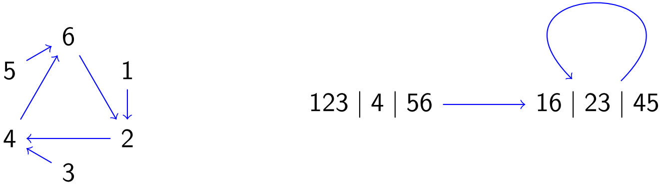


s^{-1}

Kernel space, I

- $\mathsf{P}(\Omega) := \{\text{partitions of } \Omega\}$.
- $\mathsf{P}_k(\Omega) := \{k\text{-part partitions of } \Omega\}$.
- For a map $s : \Omega \to \Omega$, define

$$\begin{array}{rccc} \tilde{s}: & \mathsf{P}(\Omega) & \to & \mathsf{P}(\Omega) \\ & \Pi & \mapsto & \{s^{-1}(\pi) \mid \pi \in \Pi\} \setminus \{\emptyset\}. \end{array}$$



s $\tilde{s}$

Kernel space, II

Let S be a transformation semigroup on Ω.

- S induces a transformation semigroup $\tilde{S} := \{\tilde{s} \mid s \in S\}$ on $\mathrm{P}(\Omega)$. We call $\tilde{S}$ the kernel space of S.
- S is k-kernel homogeneous if for all $\Pi, \Pi' \in \mathrm{P}_k(\Omega)$ there exists $s \in S$ such that $\tilde{s}(\Pi) = \Pi'$.

Theorem (Wu-Z.,2018+)

If $k \leq \ell \leq \frac{|\Omega|}{2}$, then

$$\#\,\mathrm{wcc}(\Gamma_{\tilde{S}}[\binom{\Omega}{k}]) \leq \#\,\mathrm{wcc}(\Gamma_{\tilde{S}}[\binom{\Omega}{\ell}]).$$

Moreover,

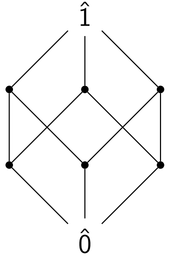
$$\ell\text{-kernel homogeneous} \Rightarrow k\text{-kernel homogeneous.}$$

Lattice

A lattice is a partially ordered set such that for every two elements x, y

- (join) $x \vee y :=$ unique $\min\{z \mid x \leq z, y \leq z\}$ and
- (meet) $x \wedge y :=$ unique $\max\{z \mid z \leq x, z \leq y\}$ exists.

We say that x is covered by y if there is no z such that $x < z < y$ and $x < y$, denoted by $x \lessdot y$.

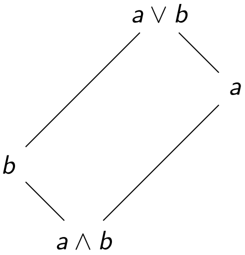


atoms

Geometric lattice

A geometric lattice is an atomistic semimodular lattice.

- atomistic: $\hat{1} = \bigvee\{\text{atoms}\}$
- semimodular: $a \wedge b \lessdot b \Rightarrow a \lessdot a \vee b$.



Example

- Boolean lattice of Ω: 2^{Ω} with set-inclusion.
- Partition lattice of Ω: $\mathrm{P}(\Omega)$ with refinement.
- Projective geometry on $\mathbb{F}_q^n$: the set of all subspaces of $\mathbb{F}_q^n$ with set-inclusion.

Inclusion operator, I

Let L be a geometric lattice.

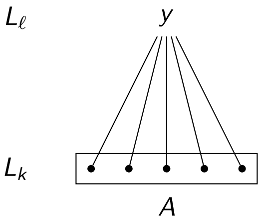
- The rank $r_L(x)$ of an element x in L is defined to be the common length of saturated chains from $\hat{0}$ to x.

Given k and ℓ such that $k \leq \ell \leq r_L(\hat{1})$,

- $L_k := \{x \in L \mid r_L(x) = k\}$.
- Inclusion operator $\zeta_L^{k,\ell} : \mathbb{C}^{L_k} \to \mathbb{C}^{L_\ell}$ is a linear map such that

$$\left(\zeta_L^{k,\ell}(f)\right)(y) = \sum_{x \in A} f(x)$$

for all $f \in \mathbb{C}^{L_k}$ and $y \in L_\ell$.



Inclusion operator, II

Let Ω be a finite set. Let k and ℓ be two positive integers with $k \leq \ell$.

Theorem

If $k + \ell \leq |\Omega|$, *then* $\zeta^{k,\ell}_{2^\Omega}$ *is injective.*

Theorem (Kung, 1993)

If $\ell \leq \frac{|\Omega|}{2}$, *then* $\zeta^{k,\ell}_{\mathrm{P}(\Omega)}$ *is injective.*

Theorem (Kantor, 1974)

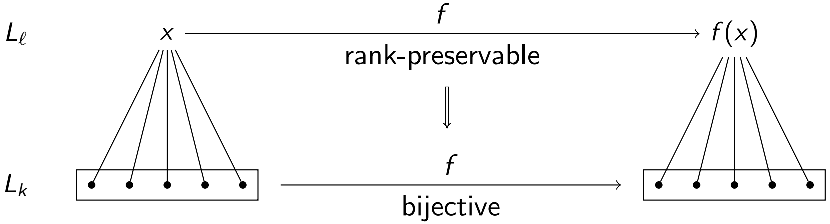
Let L *be the projective geometry on* $\mathbb{F}_q^n$. *If* $k + \ell \leq n$, *then* $\zeta^{k,\ell}_L$ *is injective.*

Conjecture (Kung, 1993)

Let L *be a finite geometric lattice. If* $\ell \leq \frac{|\Omega|}{2}$, *then* $\zeta^{k,\ell}_L$ *is injective.*

Hereditary endomorphism

Let L be a finite geometric lattice. We call $f \in \mathrm{End}(L)$ an (ℓ, k)-hereditary endomorphism if $r_L(x) = r_L(f(x))$ ensures that f induces a bijection from $\{y \mid y \in L_k, y \leq x\}$ to $\{z \mid z \in L_k, z \leq f(x)\}$.



- every element in $\overline{S}$ is (ℓ, k)-hereditary on 2^Ω;
- every element in $\tilde{S}$ is (ℓ, k)-hereditary on $P(\Omega)$.

Lemma

Let L be a geometric lattice. Let k and ℓ be two positive integers. Let S be a transformation semigroup on L such that every map in S is an (ℓ, k)-hereditary endomorphism. If $\zeta_L^{k,\ell}$ is injective then

$$\#\,\mathsf{wcc}(\Gamma_S[L_k]) \leq \#\,\mathsf{wcc}(\Gamma_S[L_\ell]).$$

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Proof.

- Let $W \subseteq \mathbb{C}^{L_k}$ be the set of all functions which are constant on each weakly connected component of $\Gamma_S[L_k]$.

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- $\zeta_L^{k,\ell}(W) \subseteq V$.

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- $\zeta_L^{k,\ell}(W) \subseteq V$.
- $\dim(W) \leq \dim(V)$.

$\square$

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- $\#\operatorname{wcc}(\Gamma_S[L_k]) = \dim(W) \leq \dim(V)$.

□

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- $\zeta_L^{k,\ell}(W) \subseteq V$.
- $\#\operatorname{wcc}(\Gamma_S[L_k]) = \dim(W) \leq \dim(V) = \#\operatorname{wcc}(\Gamma_S[L_\ell])$.

□

Two similar results

Let S be a transformation semigroup on Ω. Let A be a subset of Ω. The stabiliser permutation group of (S, A) is the permutation group $S_A := \{g|_A \mid g \in S, A\overline{g} = A\}$ on A.

Theorem (Wu-Z., 2018+)

Let $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$. If $k + \ell \leq |\Omega| - 1$ and $k \leq \ell$, then

$$\#\textit{orbits of } S_A \leq \#\textit{orbits of } S_B.$$

Theorem (Wu-Z., 2018+)

Let L be the projective geometry on $\mathbb{F}_q^n$ and $S \subseteq \mathrm{Mat}_n(\mathbb{F}_q)$. If $k + \ell \leq n$ and $k \leq \ell$, then

$$\#\,\mathrm{wcc}\,\Gamma_S[L_k] \leq \#\,\mathrm{wcc}\,\Gamma_S[L_\ell].$$

Thank you