On perfect 2-colorings of infinite circulant graphs with a continuous set of odd distances

Olga Parshina

Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia
Institut Camille Jordan UCBL1, Lyon, France

joint work with Maria Lisitsyna
On equitable 2-partitions of infinite circulant graphs with a continuous set of odd distances

Olga Parshina

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joint work with Maria Lisitsyna
Let $G = (V, E)$ be a graph, and $k$ be a positive integer.

- Consider a **partition** of the set of vertices $V = V_1 \cup V_2 \cup \cdots V_k$ of the graph $G$ into $k$ pairwise disjoint subsets (**$k$-partition**).

- A $k$-partition is called **equitable** if every vertex of $V_i$ has exactly $m_{ij}$ neighbors of $V_j$.

- The matrix $M = (m_{ij})$ is the **quotient matrix** of equitable $k$-partition.

\[
\begin{pmatrix}
g & r & b \\
1 & 3 & 0 \\
1 & 0 & 2 \\
0 & 1 & 2 \\
\end{pmatrix}
\]
Perfect colorings

Let $G = (V, E)$ be a graph, and $k$ be a positive integer

- A **coloring** of the vertices of the graph $G$ with $k$ colors (**$k$-coloring**) is a map
  $$\varphi : V \rightarrow \{1, 2, \ldots, k\}$$

- A vertex $v \in V$ is said to be of **color** $s$, if $\varphi(v) = s$

- A $k$-coloring is called **perfect** with parameter matrix $M = (m_{ij})$ if for each vertex of color $i$ the number of adjacent vertices of color $j$ is equal to $m_{ij}$.

$$M = \begin{pmatrix}
g & r & b \\
1 & 3 & 0 \\
1 & 0 & 2 \\
0 & 1 & 2
\end{pmatrix}$$
Perfect colorings

- A coloring \( \varphi \) is periodic if there exists \( t \in \mathbb{N} \) s.t.
  \( \varphi(v_i) = \varphi(v_{i+t}) \) for every vertex \( v_i \)
- The sequence \( T = [\varphi(v_i) \varphi(v_{i+1}) \varphi(v_{i+2}) \ldots \varphi(i+t-1)] \), is called the period of the coloring \( \varphi \)
- The length of the period \( T \) is \( t \)

Example

Cycle graph

\[ T = [bbr] \]
A **finite circulant of length** $t$ is a pseudograph $Ci_t(d_1, d_2, ..., d_n) = (V, E)$ with the set of vertices $V = \mathbb{Z}_t$ and the set of edges $E = \{(i, i \pm dj \mod t) | i \in \mathbb{Z}_t, j = 1, ..., n\}$.

**Examples**

$Ci_6(1, 2, 4)$

$Ci_4(1, 2, 3)$
Circulants

An infinite circulant graph is a graph $Ci_\infty(d_1, d_2, \ldots, d_n) = (V, E)$ with the set of vertices $V = \mathbb{Z}$ and the set of edges $E = \{(i, i \pm d_j) | i \in \mathbb{Z}, j = 1, \ldots, n\}$.

Example

$Ci_\infty(1, 2, 4)$


Circulants

A perfect coloring of a finite circulant graph $Ci_t(d_1, d_2, ..., d_n)$ induce the perfect coloring of a graph $Ci_\infty(d_1, d_2, ..., d_n)$ with the same parameter matrix.

Example

$Ci_\infty(1, 2, 4), \ T = [rgbgr]$ 

$M = \begin{pmatrix} 
1 & 3 & 2 \\
3 & 0 & 3 \\
2 & 3 & 1 
\end{pmatrix}$
Circulants

Let $n \in \mathbb{N}$. We consider the following infinite graphs:

- $Ci_\infty(1, 2, 3, \ldots, n)$
- $Ci_\infty(1, 3, 5 \ldots, 2n - 1)$

Examples

\[ Ci_\infty(1, 3) \]

\[ Ci_\infty(1, 2, 3) \]
Perfect colorings of infinite path graph

The infinite path graph is $C_{i\infty}(1)$

Proposition

For every $k \in \mathbb{N}$ the list of perfect colorings of $C_{i\infty}(1)$ is exhausted by colorings with the following periods:

- $[123...k]$
- $[123...(k - 1)k(k - 1)...32]$
- $[123...(k - 1)k(k - 1)...321]$
- $[123...(k - 1)kk(k - 1)...321]$

Let $k$ and $n$ be positive integers. The set of perfect $k$-colorings of the graph $Ci_{\infty}(1, 2, 3, \ldots, n)$ consists of all perfect $k$-colorings of graphs $Ci_t(1, 2, 3, \ldots, n)$ for $t = 2n, 2n + 1, 2n + 2$, and perfect colorings of the infinite path graph.

P. (2014)

The conjecture is true in case $k = 2$, $n \in \mathbb{N}$.

Lisitsyna, P. (2017)

The conjecture is true in case $n = 2$, $k \in \mathbb{N}$. 
Conjecture

Let \( k \) and \( n \) be positive integers. The set of perfect \( k \)-colorings of the graph \( Ci_{\infty}(1, 3, 5, ..., 2n - 1) \) consists of all perfect \( k \)-colorings of graphs \( Ci_t(1, 3, 5, ..., 2n - 1) \) for \( t = 4n, 4n \pm 2 \), and perfect colorings of the infinite path graph.

The conjecture is true for \( k = 2 \) and arbitrary positive integer \( n \).
Theorem [Lisitsyna, P.]

Let $n$ be a positive integer. The set of perfect 2-colorings of the graph $C_{i\infty}(1, 3, 5, ..., 2n - 1)$ consists of all perfect 2-colorings of graphs $C_{i_t}(1, 3, 5, ..., 2n - 1)$ for $t = 4n, 4n \pm 2$, and perfect colorings of the infinite path graph.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a + b = c + d = 2n \implies M = \begin{pmatrix} 2n - b & b \\ c & 2n - c \end{pmatrix}$$

Parameters $b$ and $c$ are the outer degrees of colors 1 and 2 respectively.
Sketch of the proof
Sketch of the proof
Sketch of the proof
Sketch of the proof

\[ \bullet, \circ \in \{ \color{red}\bullet, \circ \} \]
Sketch of the proof
Sketch of the proof

\[
\begin{align*}
\begin{pmatrix} c & b \\ c & b \end{pmatrix} & \quad \text{red blue} \\
\begin{pmatrix} c & b \\ c & b \end{pmatrix} & \quad c + b = 2n
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix} c + 1 & b \\ c & b + 1 \end{pmatrix} & \quad \text{red blue} \\
\begin{pmatrix} c & b \\ c & b - 1 \end{pmatrix} & \quad c + b = 2n + 1
\end{align*}
\]
\[ b + c = 2n \]

The coloring is periodic with the length of period \( t = 4n \).

Example, \( n = 2 \)

\[ Ci_8(1, 3) \]
$$b + c = 2n + 1$$
Suppose, that the length of a period of the coloring is \( t = 2n + 1 \)

Example, \( n=2 \)
\[ b + c = 2n + 1 \]

Example, \( n=2 \)
$b + c = 2n + 1$

Example, $n=2$
\[ b + c = 2n + 1 \]

Example, \( n=2 \)
$b + c = 2n + 1$

Example, $n=2$

$(2n - 1)$-chain
Example, $n=2$
\[ b + c = 2n + 1 \]

Example, \( n=2 \)
\[ b + c = 2n + 1 \]

Example, \( n=2 \)
$b + c = 2n + 1$

Example, $n=2$
$b + c = 2n + 1$

Example, $n=2$
\[ b + c = 2n + 1 \]

\[ \#\text{red} = \#\text{blue} \Rightarrow b = c \]
\[ b + c = 2n + 1 \]

The coloring is periodic with the length of period \( t = 2n + 1 \).

Example, \( n=2 \)

\[ Ci_{10}(1, 3) \]

\[ Ci_{\infty}(1, 3) \]
$b + c = 2n - 1$

The coloring is periodic with the length of period $t = 2n - 1$.

Example, $n = 2$

$Ci_6(1, 3)$
Theorem [Lisitsyna, P.]

Let $n$ be a positive integer. The set of perfect 2-colorings of the graph $C_{i\infty}(1, 3, 5, \ldots, 2n - 1)$ consists of all perfect 2-colorings of graphs $C_{i\,t}(1, 3, 5, \ldots, 2n - 1)$ for $t = 4n, 4n \pm 2$, and perfect colorings of the infinite path graph.
Conjecture on $Ci_{\infty}(1, 2, 3, \ldots, n)$

Let $k$ and $n$ be positive integers. The set of perfect $k$-colorings of the graph $Ci_{\infty}(1, 2, 3, \ldots, n)$ consists of all perfect $k$-colorings of graphs $Ci_t(1, 2, 3, \ldots, n)$ for $t = 2n, 2n + 1, 2n + 2$, and perfect colorings of the infinite path graph.

Conjecture on $Ci_{\infty}(1, 3, 5, \ldots, 2n − 1)$

Let $k$ and $n$ be positive integers. The set of perfect $k$-colorings of the graph $Ci_{\infty}(1, 3, 5, \ldots, 2n − 1)$ consists of all perfect $k$-colorings of graphs $Ci_t(1, 3, 5, \ldots, 2n − 1)$ for $t = 4n, 4n \pm 2$, and perfect colorings of the infinite path graph.
Thank you for your attention!