On perfect 2-colorings of infinite circulant graphs with a continuous set of odd distances

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joint work with Maria Lisitsyna

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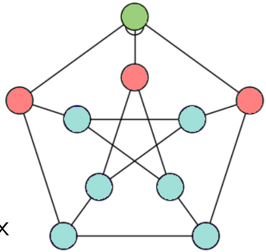
Equitable partitions

Let $G = (V, E)$ be a graph, and k be a positive ir



- Consider a *partition* of the set of vertices $V = V_1 \cup V_2 \cup \cdots V_k$ of the graph G into k pairwise disjoint subsets (*k-partition*)

- A k-partition is called *equitable* if every vertex of V_i has exactly m_{ij} neighbors of V_j
- The matrix $M = (m_{ij})$ is the *quotient matrix* of equitable k-partition

$$M = \begin{array}{c} \begin{matrix} g & r & b \end{matrix} \\ \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$$

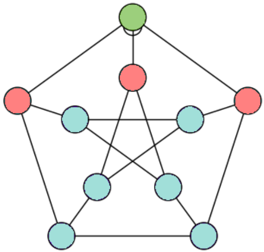
Perfect colorings

Let $G = (V, E)$ be a graph, and k be a positive integer

- A *coloring* of the vertices of the graph G with k colors (*k-coloring*) is a map $\varphi : V \to \{1, 2, ..., k\}$

- A vertex $v \in V$ is said to be of *color* s, if $\varphi(v) = s$

- A k-coloring is called *perfect* with parameter matrix $M = (m_{ij})$ if for each vertex of color i the number of adjacent vertices of color j is equal to m_{ij}.



$$M = \begin{array}{c} \begin{array}{ccc} g & r & b \end{array} \\ \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$$

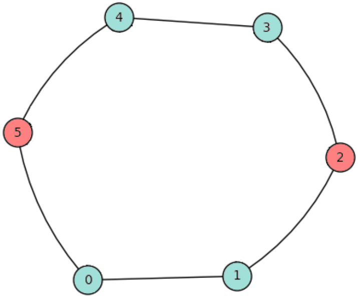
Perfect colorings

- A coloring φ is *periodic* if there exists $t \in \mathbb{N}$ s.t. $\varphi(v_i) = \varphi(v_{i+t})$ for every vertex v_i
- The sequence $T = [\varphi(v_i)\varphi(v_{i+1})\varphi(v_{i+2})...\varphi(_{i+t-1})]$, is called the *period* of the coloring φ
- The length of the period T is t

Example

Cycle graph

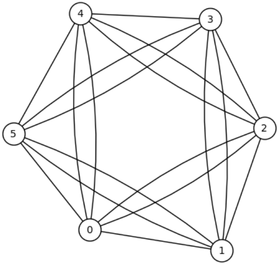
$T = [bbr]$



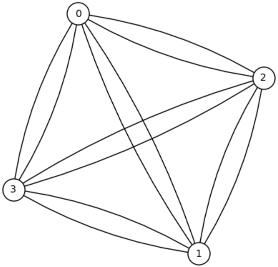
Circulants

- A *finite circulant of length t* is a pseudograph $Ci_t(d_1, d_2, ..., d_n) = (V, E)$ with the set of vertices $V = \mathbb{Z}_t$ and the set of edges $E = \{(i, i \pm d_j \bmod t) | i \in \mathbb{Z}_t, j = 1, ..., n\}$

Examples



$Ci_6(1, 2, 4)$

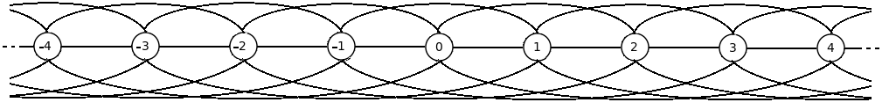


$Ci_4(1, 2, 3)$

Circulants

- An *infinite circulant graph* is a graph $Ci_\infty(d_1, d_2, ..., d_n) = (V, E)$ with the set of vertices $V = \mathbb{Z}$ and the set of edges $E = \{(i, i \pm d_j) | i \in \mathbb{Z}, j = 1, ..., n\}$.

Example

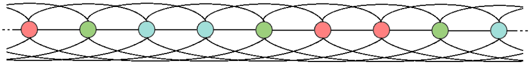


$Ci_\infty(1, 2, 4)$

Circulants

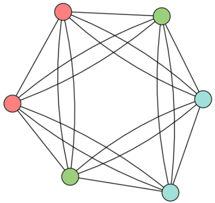
A perfect coloring of a finite circulant graph $Ci_t(d_1, d_2, \dots, d_n)$ induce the perfect coloring of a graph $Ci_\infty(d_1, d_2, \dots, d_n)$ with the same parameter matrix.

Example



$Ci_\infty(1,2,4),\ T = [rgbbgr]$

$$M = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} r & g & b \end{array} \\ \begin{pmatrix} 1 & 3 & 2 \\ 3 & 0 & 3 \\ 2 & 3 & 1 \end{pmatrix} \end{array}$$



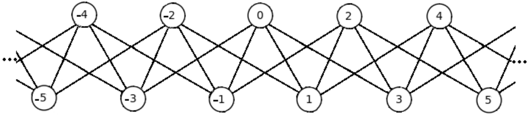
$Ci_6(1,2,4)$

Circulants

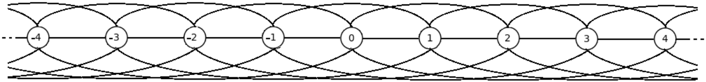
Let $n \in \mathbb{N}$. We consider the following infinite graphs:

- $Ci_\infty(1, 2, 3, \ldots, n)$
- $Ci_\infty(1, 3, 5 \ldots, 2n-1)$

Examples



$Ci_\infty(1, 3)$



$Ci_\infty(1, 2, 3)$

Perfect colorings of infinite path graph

The infinite path graph is $Ci_\infty(1)$

Proposition

For every $k \in \mathbb{N}$ the list of perfect colorings of $Ci_\infty(1)$ is exhausted by colorings with the following periods:

- $[123...k]$
- $[123...(k-1)k(k-1)...32]$
- $[123...(k-1)k(k-1)...321]$
- $[123...(k-1)kk(k-1)...321]$

Conjecture, P. (2015)

Let k and n be positive integers. The set of perfect k-colorings of the graph $Ci_\infty(1, 2, 3, \ldots, n)$ consists of all perfect k-colorings of graphs $Ci_t(1, 2, 3, \ldots, n)$ for $t = 2n, 2n + 1, 2n + 2$, and perfect colorings of the infinite path graph.

P. (2014)

The conjecture is true in case $k = 2$, $n \in \mathbb{N}$.

Lisitsyna, P. (2017)

The conjecture is true in case $n = 2$, $k \in \mathbb{N}$.

Conjecture

Let k and n be positive integers. The set of perfect k-colorings of the graph $Ci_\infty(1, 3, 5, ..., 2n-1)$ consists of all perfect k-colorings of graphs $Ci_t(1, 3, 5, ..., 2n-1)$ for $t = 4n, 4n \pm 2$, and perfect colorings of the infinite path graph.

The conjecture is true for $k = 2$ and arbitrary positive integer n.

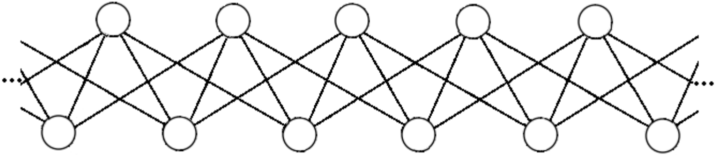
Theorem [Lisitsyna, P.]

Let n be a positive integer. The set of perfect 2-colorings of the graph $Ci_\infty(1, 3, 5, ..., 2n-1)$ consists of all perfect 2-colorings of graphs $Ci_t(1, 3, 5, ..., 2n-1)$ for $t = 4n, 4n \pm 2$, and perfect colorings of the infinite path graph.

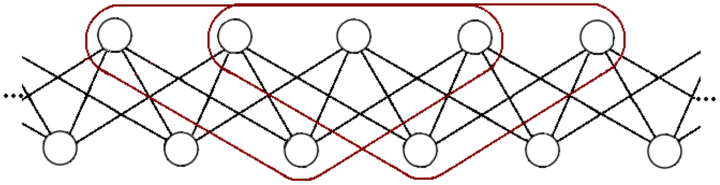
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a + b = c + d = 2n \implies M = \begin{pmatrix} 2n-b & b \\ c & 2n-c \end{pmatrix}$$

Parameters b and c are the *outer degrees* of colors 1 and 2 respectively

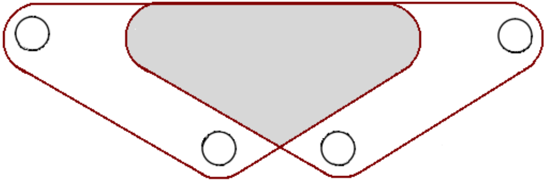
Sketch of the proof



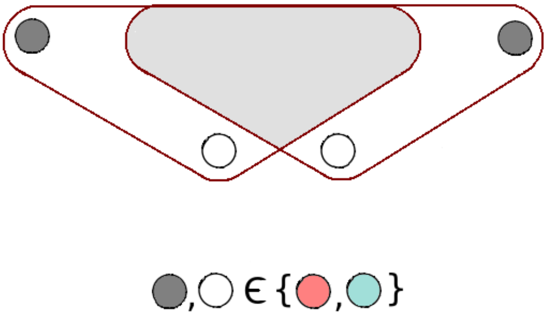
Sketch of the proof



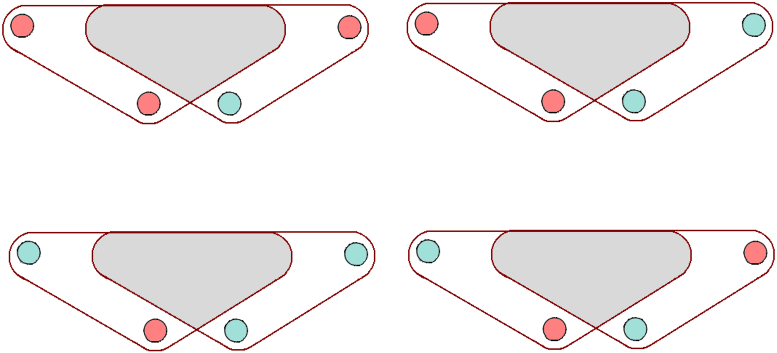
Sketch of the proof



Sketch of the proof



Sketch of the proof



Sketch of the proof

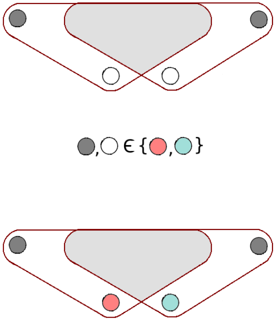
$$\begin{array}{c} \begin{matrix} \text{red} & \text{blue} \end{matrix} \\ \begin{pmatrix} c & b \\ c & b \end{pmatrix} \end{array} \qquad \begin{array}{c} \begin{matrix} \text{red} & \qquad \text{blue} \end{matrix} \\ \begin{pmatrix} c+1 & b \\ c & b+1 \end{pmatrix} \end{array}$$

$$c + b = 2n \qquad c + b = 2n - 1$$

$$\begin{pmatrix} c & b \\ c & b \end{pmatrix} \qquad \begin{pmatrix} c-1 & b \\ c & b-1 \end{pmatrix}$$

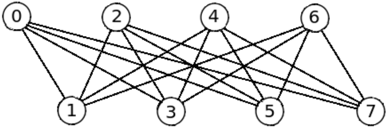
$$c + b = 2n \qquad c + b = 2n + 1$$

$b + c = 2n$



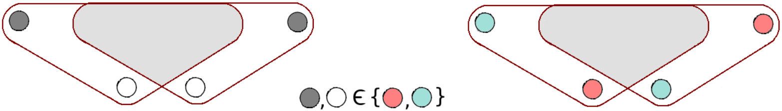
The coloring is periodic with the length of period $t = 4n$.

Example, $n = 2$

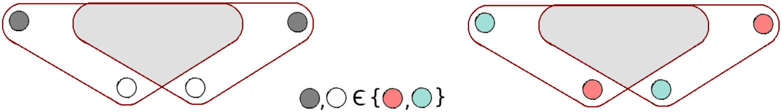


$Ci_8(1, 3)$

$b + c = 2n + 1$

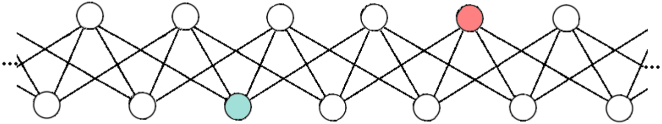


$b + c = 2n + 1$



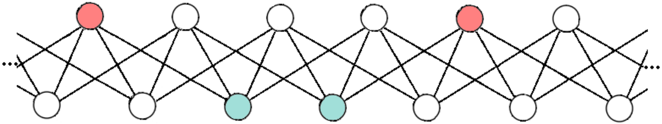
Suppose, that the length of a period of the coloring is $t = 2n + 1$

Example, n=2



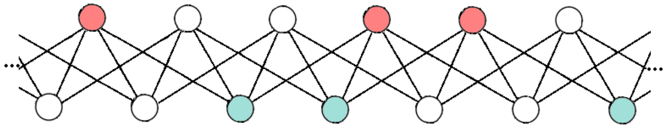
$b + c = 2n + 1$

Example, n=2



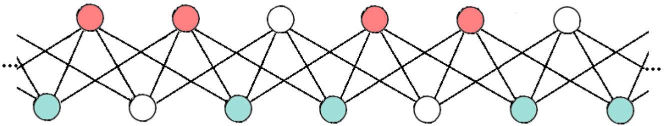
$b + c = 2n + 1$

Example, n=2



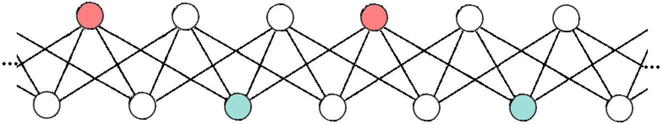
$b + c = 2n + 1$

Example, n=2



$b + c = 2n + 1$

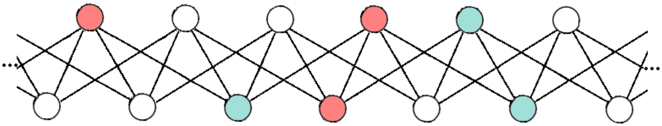
Example, n=2



$(2n-1)$-chain

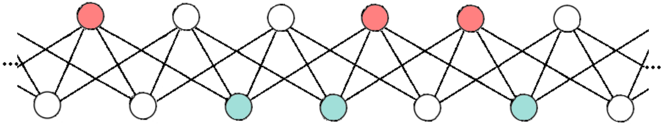
$b + c = 2n + 1$

Example, n=2



$b + c = 2n + 1$

Example, n=2



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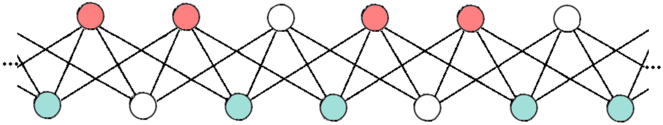
Example, n=2



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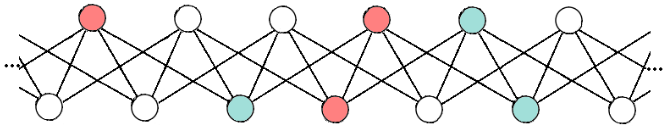
Example, n=2



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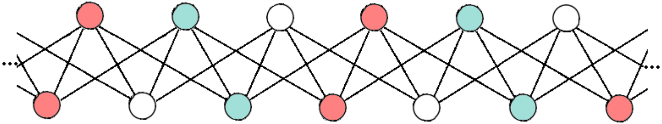
Example, n=2



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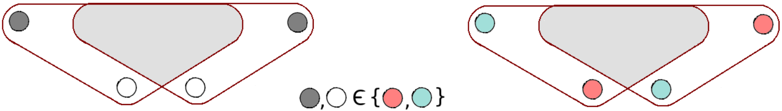
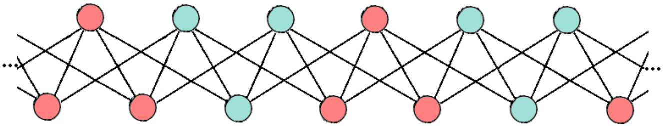


Example, n=2

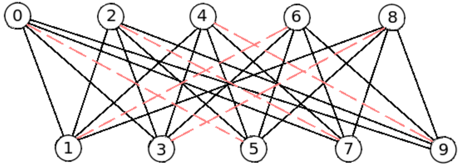


$$\#red = \#blue \Rightarrow b = c$$

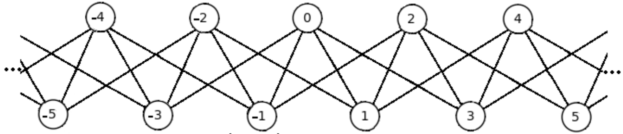
$b + c = 2n + 1$

The coloring is periodic with the length of period $t = 2n + 1$.

Example, n=2

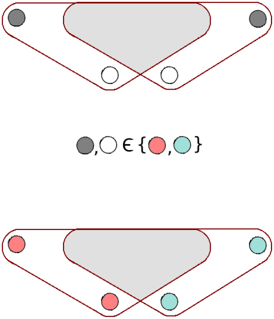


$Ci_{10}(1, 3)$



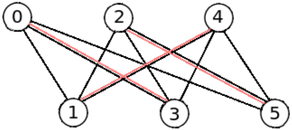
$Ci_{\infty}(1, 3)$

$b + c = 2n - 1$



The coloring is periodic with the length of period $t = 2n - 1$.

Example, $n = 2$



$Ci_6(1,3)$

Theorem [Lisitsyna, P.]

Let n be a positive integer. The set of perfect 2-colorings of the graph $Ci_\infty(1, 3, 5, ..., 2n-1)$ consists of all perfect 2-colorings of graphs $Ci_t(1, 3, 5, ..., 2n-1)$ for $t = 4n, 4n \pm 2$, and perfect colorings of the infinite path graph.

Conjecture on $Ci_\infty(1, 2, 3, \ldots, n)$

Let k and n be positive integers. The set of perfect k-colorings of the graph $Ci_\infty(1, 2, 3, \ldots, n)$ consists of all perfect k-colorings of graphs $Ci_t(1, 2, 3, \ldots, n)$ for $t = 2n, 2n + 1, 2n + 2$, and perfect colorings of the infinite path graph.

Conjecture on $Ci_\infty(1, 3, 5, ..., 2n - 1)$

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Thank you for your attention!