PC-polynomial of graph and its largest root

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Outline

1 Introduction

2 Random graphs and average value of $\beta(G)$

3 Bounds on $\beta(G)$ and Nordhaus—Gaddum type inequalities

Partially commutative monoid

Initial problem (with Mikhail Novikov)

What does the growth rate of partially commutative Lie algebra equal?

[Poroshenko, 2011]: the linear basis of partially commutative Lie algebra

[Duchamp & Krob, 1992]: the growth rate of partially commutative Lie algebra equals the growth rate of partially commutative monoid

$G = G(X, E)$: simple finite graph
$M(G) = M\langle X \mid ab = ba,\ (a, b) \in E\rangle$: partially commutative monoid [Cartier & Foata, 1969]

Example

$X = \{a, b, c\}$, $E = \{(a, b), (b, c)\}$. Then $bbac = abcb$ as
$b\underline{ba}c = ba\underline{bc} = \underline{ba}cb = abcb$

PC-polynomial of graph

m_n: the number of all pairwise different words of the length n in $M(G)$
$c_i(G)$: the number of distinct cliques in G of size i
$t_0 = \omega(G)$: the size of a maximal clique in G

Theorem 1 [Cartier & Foata, 1969; Fisher, 1989]

The numbers m_n, $n \geq 1$, satisfy the recurrence relation

$$m_n = c_1(G)m_{n-1} - c_2(G)m_{n-2} + \ldots + (-1)^{t_0+1}c_{t_0}(G)m_{n-t_0} \quad (1)$$

with initial data $m_0 = 1$, $m_{-1} = \ldots = m_{-t_0+1} = 0$

Define PC-polynomial on graph G as a characteristic polynomial of (1):

$$PC(G,x) = x^{t_0} - c_1(G)x^{t_0-1} + c_2(G)x^{t_0-2} + \ldots + +(-1)^{t_0}c_{t_0}(G) \quad (2)$$

Examples

a) $PC(\bar{K}_n, x) = x - n$; b) $PC(T_n, x) = x^2 - nx + (n-1)$, T_n: tree
c) $PC(K_{q_1,\ldots,q_s}) = (x-q_1)\ldots(x-q_s)$, d) $PC(K_n, x) = (x-1)^n$

Clique-type polynomials on graph

$PC(G,x) = \sum_{k=0}^{\omega(G)} (-1)^k c_k(G) x^{\omega(G)-k}$ — PC-polynomial [V.G.]

$D(G,x) = 1 + \sum_{k=1}^{\omega(G)} (-1)^k c_k(G) x^k$ — dependence polynomial [Fisher & Solow, 90']

$C(G,x) = 1 + \sum_{k=1}^{\omega(G)} c_k(G) x^k$ — clique polynomial [Hoede & Li, 94']

$I(G,x) = 1 + \sum_{k=1}^{\alpha(G)} s_k(G) x^k$ — independence pol. [Gutman & Harary, 83']

$\mu(G,x) = 1 + \sum_{k=1}^{\nu(G)} (-1)^k m_k x^{n-2k}$ — matching polynomial [Hosoya, 1971]

$$D(G,x) = x^{\omega(G)} PC(G,1/x), \quad C(G,x) = D(G,-x)$$
$$I(G,x) = C(\bar{G},x), \quad \mu(G,x) = x^n I(L(G),-x^{-2})$$

Definition of $\beta(G)$

z_0: maximal on modulus (complex) root of $PC(G,x)$, $\beta(G) = |z_0|$

Theorem 2 [Fisher, Solow, 90'; Goldwurm, Santini, 00'; Csikvári, 13']

The number $\beta(G)$ is a root of $PC(G,x)$ and modulus of any other root of $PC(G,x)$ is less than $\beta(G)$

Key equality

$$D(G,x) = D(G \setminus v, x) - xD(G[N(v)], x), \quad v \in V(G)$$

Corollary 1 [Goldwurm & Santini, 2000; Csikvári, 2013]

The growth rate of a partially commutative monoid $M(G)$ equals $\beta(G)$, i.e. $\beta(G) = \lim_{n\to\infty} \sqrt[n]{m_n(G)}$

Examples of $\beta(G)$

Examples

$$
\begin{array}{lcl}
PC(K_n,x) = (x-1)^n & \Rightarrow & \beta(K_n) = 1 \\
PC(\bar{K}_n,x) = x-n & \Rightarrow & \beta(\bar{K}_n) = n \\
PC(K_{n_1,n_2}) = (x-n_1)(x-n_2) & \Rightarrow & \beta(K_{n_1,n_2}) = \max\{n_1,n_2\} \\
PC(T_n,x) = x^2 - nx + (n-1) & \Rightarrow & \beta(T_n) = n-1
\end{array}
$$

Mantel's theorem, 1907

Maximal number of edges in a graph on n vertices and without triangles equals $[n^2/4]$

Proof [Hajiabolhassan, 98]. $PC(G,x) = x^2 - nx + k$ where $k = |E(G)|$

$\beta(G)$ is a real root of $PC(G,x) \ \Rightarrow\ D = n^2 - 4k \geq 0 \ \ \Rightarrow\ k \leq n^2/4$

The bound is reached on $K_{[n/2],n-[n/2]}$

Remark

Turán's theorem follows from $C(G,x) \leq \left(1 + \frac{nx}{w}\right)^w$, $x > 0$ [Galvin, 2009]

Problems

Denote the set of all graphs with n vertices and k edges as $G(n,k)$,

$$\beta_-(n,k) = \min_{G \in G(n,k)} \beta(G), \quad \beta_+(n,k) = \max_{G \in G(n,k)} \beta(G)$$

1. Find the values $\beta_\pm(n,k)$ and the graphs on which they are reached

2. Nordhaus—Gaddum type problem. Find exact bounds for $\beta(G) + \beta(\bar{G})$ and $\beta(G)\beta(\bar{G})$ in the terms of $n = |V(G)|$
($2\sqrt{n} \leq \chi(G) + \chi(\bar{G}) \leq n+1, \quad n \leq \chi(G)\chi(\bar{G}) \leq \left(\frac{n+1}{2}\right)^2$)

3. Find the average value of $\beta(G)$ on graphs with n vertices

Some known results

A claw-free graph is a graph without induced $K_{1,3}$-subgraphs

Theorem 3 [Chudnovsky & Seymour, 2007]

Let $\bar{G}$ be a claw-free graph, then all roots of $PC(G,x)$ are real

Conjecture 1 [Alavi, Malde, Schwenk, Erdös, 1987]

Let $\bar{G}$ be a tree, then the numbers $c_i = c_i(G)$ satisfy
$c_0 \leq \ldots \leq c_{k-1} \leq c_k \geq c_{k+1} \geq \ldots \geq c_n$ (unimodality of $PC(G,-x)$)

Theorem 4 [Brown & Nowakowski, 2005]

For almost all graphs, $PC(G,x)$ has a complex root

PC–polynomial of random graph

For random graph $G_{n,p}$ with n vertices and edge probability p define

$$PC(G_{n,p},x) = x^n - \binom{n}{1}x^{n-1} + \binom{n}{2}px^{n-2} - \ldots + (-1)^k\binom{n}{k}p^{\frac{k(k-1)}{2}}x^k + \ldots + (-1)^n p^{\frac{n(n-1)}{2}}$$

$$PC(G_{2,p},x) = x^2 - 2x + p = (x - (1-\sqrt{1-p}))(x - ({\color{red}1+\sqrt{1-p}}))$$

Theorem 5 [Brown, Dilcher, Manna, 2013]

Let $p \in (0,1)$. All roots of $PC(G_{n,p},x)$ are simple, separated and real.
Proof (real-rootedness).

$$x^n PC(G_{n,p},-1/x) = \sum_{k=0}^{n}\binom{n}{k}q^{k^2}y^k \quad \text{where } \sqrt{p}=q,\ x=yq$$

The polynomial $\sum_{k=0}^{n}\binom{n}{k}y^k = (1+y)^n$ has only real roots, thus all roots of $\sum_{k=0}^{n}\binom{n}{k}q^{k^2}y^k$ are real provided $|q| \le 1$ [Laguerre, 1882]

Results on $PC(G_{n,p}, x)$

$\beta(G_{n,p})$: maximal root of $PC(G_{n,p}, x)$

- The "random" algebra $\mathrm{As}(X, p)$ is defined as trivial deformation of the free associative algebra
- Expected values of the dimensions of homogeneous subspaces of $\mathrm{As}(X, p)$ satisfy a linear recurrence equation which characteristic polynomial is exactly $PC(G_{n,p}, x)$
- The expected growth rate of $\mathrm{As}(X, p)$ equals $\beta(G_{n,p})$
- By a change of "variables" x, p, $PC(G_{n,p}, x)$ could be done symmetric So, for odd n, $p^{\frac{n-1}{2}}$ is a middle root of $PC(G_{n,p}, x)$. All other roots of $PC(G_{n,p}, x)$ for odd n and all roots for even n could be gathered in pairs with the roots product equal p^{n-1}
- We have

$$\frac{\beta(G_{n,p})}{n} \sim 1 - \frac{p}{2} - \frac{p^2}{4} - \frac{p^3}{12} - \frac{p^4}{16} - \frac{p^5}{48} - \frac{7p^6}{288} - \frac{p^7}{96} - \frac{7p^8}{768} + O(p^9)$$

Average value of $\beta(G)$

Denote by $\beta_r(G_{n,p})$ the rth largest root of $PC(G_{n,p})$

Lemma 1

For all $p \in (0;1]$ and $r \geq 1$, there exists the limit $\lim\limits_{n\to\infty} \frac{\beta_r(G_{n,p})}{n}$

Theorem 6

The average value of $\beta(G)$ on graphs with $n \gg 1$ vertices asymptotically equals

$$\beta_{\mathrm{ev}}(n) \sim n \lim_{n\to\infty} \frac{\beta(G_{n,1/2})}{n} = \beta_0 n \approx 0.672008n$$

(β_0 from [Stanley, 1973] in counting of acyclic orientations of a digraph)

Theorem 7

Let $r > 0$. For almost all graphs with $n \gg 1$ vertices, the real roots of PC-polynomial which moduli is not less than n/r are simple, separated and their ratios over n lie in neighbourhoods of the ratios of the largest roots of $PC(G_{n,1/2}, x)$ over n

Lower bound on $\beta(G)$ and lower NG-inequlities

By computational experiments: $n - \frac{Ak}{n} \leq \beta(G) \leq n - \frac{Bk}{n}$

Theorem 8

Given a graph G with n vertices and k edges
a) [Fisher, 1989] we have $n - \frac{2k}{n} \leq \beta(G)$
b) The bound $n - \frac{2k}{n} \leq \beta(G)$ is reached if and if only G is the empty graph or complete multipartite graph with equal parts
PROOF of a). Let H equal $\bar{G}$ with all n loops. Denote by $W_s(H)$ the number of walks in H of length s. From $W_s(H) \leq m_s(G)$ we get $\rho(H) = \underline{\rho(\bar{G}) + 1 \leq \beta(G)}$. It remains to apply known bound $\rho(\bar{G}) \geq \frac{2\bar{k}}{n}$

Corollary 2

Given a graph G on n vertices, we have
a) $n + 1 \leq \beta(G) + \beta(\bar{G})$,
b) $n \leq \beta(G)\beta(\bar{G})$,
moreover, the bounds are reached if and only if $(G, \bar{G}) = (K_n, \bar{K}_n)$

The value of $\beta_-(n,k)$

Lemma 2

Let $k \leq \frac{n^2}{4}$, then $\beta_-(n,k) = \frac{n+\sqrt{n^2-4k}}{2}$. We have $\beta(G) = \beta_-(n,k)$ for $G \in G(n,k)$ if and only if G is a triangle-free graph

Theorem 9 [Fisher & Nonis, 1990]

Given a graph G with n vertices and k edges, let w be such a natural number that $(1 - \frac{1}{w-1})\frac{n^2}{2} < k \leq (1 - \frac{1}{w})\frac{n^2}{2}$, $w \geq 2$, then

$$\frac{n}{w} + \frac{1}{w}\sqrt{n^2 - \frac{2kw}{w-1}} \leq \beta_-(n,k) < \frac{n}{w} + \frac{1}{w}\sqrt{n^2 - \frac{2kw}{w-1}} + 1$$

Graphs for $\beta_-(n,k)$

In general case, we construct a supergraph G_- of $K_{a,a,\dots,a,b}$ with $w-1$ parts of a and $a>b\geq 0$, here $a=\left\lceil \frac{n}{w}+\frac{1}{w}\sqrt{n^2-2kw/(w-1)}\right\rceil$
Further, the $[k'/(w-1)]$ edges $(k'=k-((w-1)an-\binom{w}{2}a^2))$ form a triangle-free graph in each part with a vertices and we put remaining $k'-(w-1)[k'/(w-1)]$ edges anywhere

Conjecture 2

Let $k>[n^2/4]$ and G be such a graph that $\beta(G)=\beta_-(n,k)$. Then $\bar{G}$ is disconnected

Theorem 10

If Conjecture 2 holds, then $\beta_-(n,k)$=$\beta(G_-)$ for the constructed graph G_-

Graphs for $\beta_+(n,k)$

Theorem 11

Let G be such graph on n vertices that all roots of $PC(G,x)$ are real. Then $\beta \leq n - \frac{k}{n}$ (Samuelson's inequality)

Theorem 12 (Conjecture of Fisher & Nonis, 1990)

Let n, k be natural numbers, $k = \binom{d}{2} + e \leq \binom{n}{2}$ for $0 \leq e < d$. Construct a graph $G \in G(n,k)$ as follows: we add a vertex of the degree e to K_d and leave all other vertices to be isolated. Then $\beta_+(n,k) = \beta(G)$

Sketch of Proof

Step 1 [Csikvári, 2011]. There exists such a threshold graph H that $\beta(H) = \beta_+(n,k)$ (via Kelmans transformations)
Step 2. Comparison of the values of m_n for threshold graphs (via special transformations)

The upper NG-inequalities

Corollary 3

For $n \gg 1$, we have $\beta_+(n,k) \sim \sqrt{2k}/W(\frac{\sqrt{2k}}{n-\sqrt{2k}})$
($W(x)$ *is the Lambert* W*-function, the inverse to* $f(y) = ye^y$)

Corollary 4

Let $n \gg 1$. For $G \in G(n,k)$, we have
a) $\beta(G) \leq n - \frac{Bk}{n}$ for $B \approx 0.9408008$ ($n - \frac{2k}{n} \leq \beta(G)$ for all n)
b) $\beta(G) + \beta(\bar{G}) < 1.502n$
c) $\beta(G)\beta(\bar{G}) < 0.564n^2$

Statement

Let $n \gg 1$ and G be a graph from the class $A(n) = \{K_s \cup \overline{K_{n-s}}\}$, then the maximal Nordhaus—Gaddum values on $A(n)$ are the following

$$\beta(G) + \beta(\bar{G}) \approx 1.466n$$

$$\beta(G)\beta(\bar{G}) \approx 0.536n^2$$

Asymptotic borders of $\beta(G)$

Let $n = |V(G)| \gg 1$ and $x = \frac{2k}{n^2} \in (0,1)$

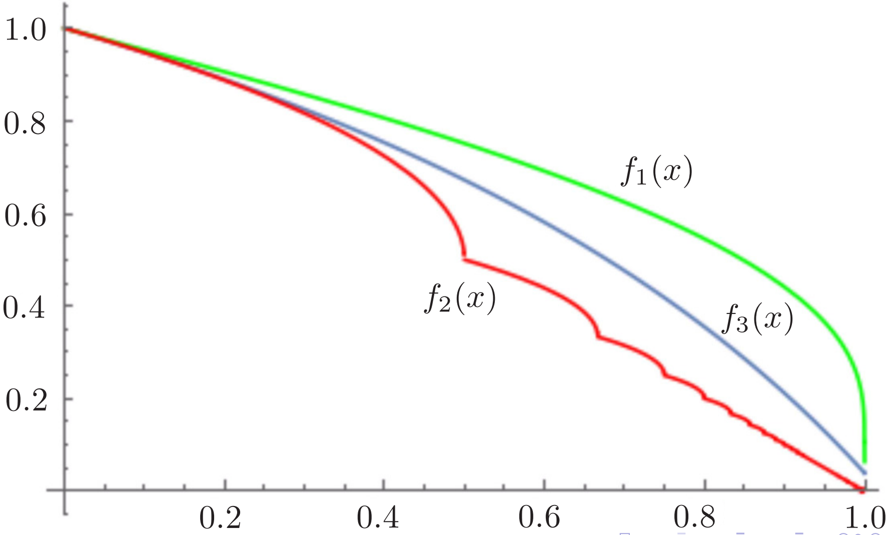
Define

$$f_1(x) = \frac{\sqrt{x}}{W(\frac{\sqrt{x}}{1-\sqrt{x}})}$$

$$f_2(x) = \frac{1}{\lceil \frac{1}{1-x} \rceil}\left(1 + \sqrt{1 - \frac{\lceil \frac{1}{1-x} \rceil x}{\lceil \frac{1}{1-x} \rceil - 1}}\right)$$

$$f_3(x) = 1 - \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{12} - \frac{x^4}{16} - \frac{x^5}{48} - \frac{7x^6}{288} - \frac{x^7}{96} - \frac{7x^8}{768}$$

Asymptotic borders of $\beta(G)$. Plot



Lovász local lemma

Theorem 13 [Scott & Sokal, 2005]

Let G be a graph on n vertices and events A_i, $i = 1, \dots, n$, be in one-to-one correspondence with vertices of G. Suppose that A_i is totally independent on the set of events $\{A_k \mid (i,k) \notin E(G)\}$. Let $P(A_i) \leq t$ for $i = 1, \dots, n$, then $P(\bigcap_{i=1}^{n} \overline{A_i}) > 0$ if and only if $t \leq 1/\beta(\bar{G})$

Corollary 5 (global Lovász local lemma)

Let G be a graph with $n \gg 1$ vertices and k edges, A_i, $i = 1, \dots, n$, are events with dependence graph G. Then $P(\bigcap_{i=1}^{n} \overline{A_i}) > 0$ if

a) $k \leq 0.2433n^2$ and $P(A_i) \leq \frac{4}{3n}$, $i = 1, \dots, n$, or

b) $k \leq \frac{n^2}{4}$ and $P(A_i) \leq \frac{1.3316}{n}$, $i = 1, \dots, n$

Is it useful?

Thank You for Your attention!

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