Hermitian adjacency spectrum of Cayley digraphs over dihedral group

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G2R2, Novosibirsk State University, August, 2018

Outline

- Introduction
- Related results on cubic Cayley graphs on dihedral group
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Definition

- For a group G and a subset $T \subseteq G$ with $1 \notin T$, the *Cayley digraph* $Cay(G, T)$ is the directed graph whose vertices are the elements of G and whose arc set is $\{(g, gt) : g \in G, t \in T\}$. The Cayley digraph $Cay(G, T)$ can be considered as an undirected graph if and only if T is inverse-closed. In general, Cayley digraph can be viewed as a mixed graph.
- The Hermitian adjacency matrix of Cayley digraph $Cay(G, T)$ is then a matrix $H \in C^{G \times G}$, whose (g, h)-entry H_{gh} is

$$H_{gh} = \begin{cases} 1, & \text{if } g^{-1}h \in T \text{ and } h^{-1}g \in T; \\ \sqrt{-1}, & \text{if } g^{-1}h \in T \text{ and } h^{-1}g \notin T; \\ -\sqrt{-1}, & \text{if } g^{-1}h \notin T \text{ and } h^{-1}g \in T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark

This definition was introduced by Xueliang Li (2015) and independently by Bojan Mohar for general mixed graphs (not restricted to Cayley digraphs) to study spectral theory of mixed graphs.

- A *mixed graph* is obtained from an undirected graph by orienting a subset of its edges. Formally, a mixed graph D is given by its *vertex-set* $V = V(D)$, the set $E_0(D)$ of *undirected edges* and the set $E_1(D)$ of *directed edges* or *arcs*. We adopt here the convention of representing two oppositely oriented arcs by an undirected edge.
- The *Hermitian adjacency matrix* of a mixed graph D is a matrix $H = H(D) \in C^{V \times V}$, whose (u, v)-entry H_{uv} is

$$H_{uv} = \begin{cases} 1, & \text{if } uv \in E_0(D); \\ \sqrt{-1}, & \text{if } uv \in E_1(D); \\ -\sqrt{-1}, & \text{if } vu \in E_1(D); \\ 0, & \text{otherwise.} \end{cases}$$

Introduction

- Two Cayley digraphs $Cay(G,S)$ and $Cay(G,T)$ are called Cayley isomorphic if $T = S^\sigma$ for some automorphism $\sigma \in Aut(G)$.
- A subset $S \subseteq G$ is a *CI-subset* of group G, if for any $T \subseteq G$, $Cay(G,S) \cong Cay(G,T)$ implies that $T = S^\sigma$ for some $\sigma \in Aut(G)$.
- A finite group G is an (m-CI, resp.) m-DCI-group if any (inverse-closed, resp.) subset S of G with $1 \notin S$ and $|S| \leq m$ is a CI-subset.
- Let HSpec(X) denote the spectrum of Hermitian adjacency matrix of a digraph X, called H-spectrum of X. A Cayley graph $Cay(G,S)$ is said to be Cay-DS if, for any $Cay(G,T)$, H-cospectral⇒isomorphic.

Problem (C.H. Li, 2002)

[a] *Completely determine m-DCI-groups for certain small values of $m \geq 3$. In particular, give a complete classification of 3-DCI-groups.*

[a] C.H. Li, On isomorphisms of finite Cayley graphs a survey, Discrete Math. 256 (2002) 301-334.

Introduction

Babai[1] had shown that D_{2p} is a CI-group, that is, all Cayley graphs on D_{2p} are CI-graphs.

Theorem (Qu & Yu, 1997)

[a] *For any $m \in \{1,2,3\}$, D_{2k} is m-DCI $\Leftrightarrow$ D_{2k} is m-CI $\Leftrightarrow 2 \nmid k$.*

[a] H. Qu and J. Yu, On isomorphisms of Cayley digraphs on dihedral groups, Australasian Journal of Combinatorics 15(1997), 213-220.

In this talk, we study the Hermitian adjacency spectrum of digraph over Dihedral group and show that when p is an odd prime, every $Cay(D_{2p}, S)$ for $|S| = 3$ is Cay-DS, and meantime present a spectral method to show D_{2p} is 3-DCI.

[1] Babai, L.: Isomorphism problem for a class of point-symmetric structures. Acta Math. Hungar., 29, 329-336 (1977)

Related results

The character table of the dihedral group D_{2n} is

n is odd	a^k	ba^k	n is even	a^k	ba^k
ψ_1	1	1	ψ_1	1	1
ψ_2	1	-1	ψ_2	1	-1
χ_h	$2\cos\frac{2\pi hk}{n}$	0	ψ_3	$(-1)^k$	$(-1)^k$
-	-	-	ψ_4	$(-1)^k$	$(-1)^{k+1}$
-	-	-	χ_h	$2\cos\frac{2\pi hk}{n}$	0

Related results

Theorem (Huang et. al, 2017)

Let S be a symmetric subset of D_{2n} such that $1 \notin S$. Then the Cayley graph $Cay(D_{2n}, S)$ has spectrum

$$\{[\lambda_i]^1; [\mu_{h1}]^2, [\mu_{h2}]^2 \mid 1 \le i \le 3 + (-1)^n; 1 \le h \le [\frac{n-1}{2}]\},$$

where $\lambda_i = \sum_{g \in S} \psi_i(g)$ for $1 \le i \le 3 + (-1)^n$ and

$$\begin{cases} \mu_{h1} + \mu_{h2} = \sum_{g \in S} \chi_h(g), \\ \mu_{h1}^2 + \mu_{h2}^2 = \sum_{g_1, g_2 \in S} \chi_h(g_1 g_2) \end{cases} \tag{1}$$

for $1 \le h \le [\frac{n-1}{2}]$.

Huang et. al focus on the cubic Cayley graph $Cay(D_{2n}, S)$, where $|S| = 3$. Since $Cay(D_{2n}, S)$ is cubic, $S = \{a^k, a^{-k}, ba^i\}$ ($k \neq \frac{n}{2}$ if n is even), or $S = \{ba^{k_1}, ba^{k_2}, ba^{k_3}\}$, or $S = \{a^{n/2}, ba^i, ba^j\}$ (n is even). Then $Cay(D_{2n}, S)$ is said to be of type-I , type-II and type-III if S possesses the form $\{a^k, a^{-k}, ba^i\}$, $\{ba^{k_1}, ba^{k_2}, ba^{k_3}\}$ and $\{a^{n/2}, ba^i, ba^j\}$, respectively.

H-spectrum of Cayley graph $Cay(D_{2n}, S)$ of type I, II and III

Let $Cay(D_{2n}, S)$ be the Cayley digraph on D_{2n} with respect to S, where $|S| = 3$. Huang et. al (2017) determined the spectrum of cubic Cayley graph $Cay(D_{2n}, S)$ as follows.

- If $Cay(D_{2n}, S)$ is of $type - I$, then $Cay(D_{2n}, S)$ has spectrum

$$\begin{cases} \{[3]^1, [1]^1; [2\cos\frac{2hk\pi}{n} \pm 1]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\}, & \text{if } n \text{ is odd;} \\ \{[3]^1, [1]^1, \lambda_3, \lambda_4; [2\cos\frac{2hk\pi}{n} \pm 1]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\}, & \text{if } n \text{ is even.} \end{cases}$$

- If $Cay(D_{2n}, S)$ is of $type - II$, then $Cay(D_{2n}, S)$ has spectrum

$$\begin{cases} \{[3]^1, [-3]^1; [\pm\sqrt{a_h(S)}]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\}, & \text{if } n \text{ is odd;} \\ \{[3]^1, [-3]^1, \lambda_3, \lambda_4; [\pm\sqrt{ah(S)}]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\}, & \text{if } n \text{ is even.} \end{cases}$$

where $\lambda_3, \lambda_4 = \pm[(-1)^{k_1} + (-1)^{k_2} + (-1)^{k_3}]$ and

$$a_h(S) = 3 + 2[\cos\frac{2h(k_1 - k_2)\pi}{n} + \cos\frac{2h(k_1 - k_3)\pi}{n} + \cos\frac{2h(k_2 - k_3)\pi}{n}].$$

H-spectrum of Cayley graph $Cay(D_{2n}, S)$ of type I, II and III

- If $Cay(D_{2n}, S)$ is of $type - III$, then $Cay(D_{2n}, S)$ has spectrum

$$\{[3]^1, [-1]^1, \lambda_3, \lambda_4; [(-1)^h \pm \sqrt{2\cos\frac{2\pi h(i-j)}{n} + 3}]^2 \mid 1 \le h \le [\frac{n-1}{2}]\}$$

where $\lambda_3 = (-1)^{\frac{n}{2}} + (-1)^i + (-1)^j$ and $\lambda_4 = (-1)^{\frac{n}{2}} + (-1)^i - (-1)^j$.

H-spectrum of circulant graph

Let $\mathbb{Z}_n$ be the cyclic group of integers module n. The irreducible character ϕ_h $(1 \leq h \leq n)$ of $\mathbb{Z}_n$ is given by

$$\phi_h(k) = e^{\frac{2\pi hk}{n} i} = \cos \frac{2\pi hk}{n} + i \sin \frac{2\pi hk}{n}, \text{ where } 0 \leq k \leq n-1.$$

Lemma (Huang et. al, 2017)

Suppose that $S \subseteq \mathbb{Z}_n \backslash \{0\}$ and $S = -S$. Then the circulant graph $Cay(\mathbb{Z}_n, S)$ has eigenvalues

$$\lambda_h = \sum_{k \in S} \phi_h(k) = \sum_{k \in S} e^{\frac{2\pi hk}{n} i} = \sum_{k \in S} \cos \frac{2\pi hk}{n},$$

where $1 \leq h \leq n$.

Cubic Cayley graphs on D_{2p}

Theorem (Huang et. al, 2017)

Let $Cay(D_{2p}, S)$ and $Cay(D_{2p}, T)$ be two cubic Cayley graphs on D_{2p}. Then the following are equivalent:

1. *HSpec($Cay(D_{2p}, S)$)=HSpec($Cay(D_{2p}, T)$).*
2. *There exists some $\lambda \in \mathbb{Z}_n^*$ and $\mu \in \mathbb{Z}_n$ such that $T = \sigma_{\lambda,\mu}(S)$, where $\sigma_{\lambda,\mu} \in Aut(D_{2p})$.*
3. *$Cay(D_{2p}, S) \cong Cay(D_{2p}, T)$.*

From above, it follows that all cubic Cayley graphs on D_{2p} are CI-graphs.

H-spectrum of Cayley digraph

- For Cayley digraph $Cay(G, T)$, define a function $\alpha : G \to \mathbb{C}$ as follows:

$$\alpha(g) = \begin{cases} 1, & \text{if } g \in T \text{ and } g^{-1} \in T; \\ \sqrt{-1}, & \text{if } g \in T \text{ and } g^{-1} \notin T; \\ -\sqrt{-1}, & \text{if } g \notin T \text{ and } g^{-1} \in T; \\ 0, & \text{otherwise.} \end{cases}$$

- According to a well known result of Babai [1] on spectrum of Cayley color graph, we can express the H-spectrum of the Cayley digraph $Cay(G, T)$ as an immediate consequence.

[1] Babai, L.: Spectra of Cayley graphs. J. Combin. Theory Ser. B, 27, 180-189 (1979)

H-spectrum of Cayley digraph

G denotes a finite group of order n whose irreducible characters are $\chi_1, \ldots, \chi_h$ with respective degrees $n_1, \ldots, n_h$ $(\sum_{i=1}^{h} n_i^2 = n)$.

Lemma

The H-spectrumof the Cayley digraph $Cay(G, T)$ is

$$HSpec(Cay(G,T)) = \{[\lambda_{i,k_i}]^{d_i} \mid 1 \leq i \leq h, 1 \leq k_i \leq d_i\}$$

where for any natural number t,

$$\sum_{k_i=1}^{d_i} \lambda_{i,k_i}^t = \sum_{g_1,\cdots,g_t \in G} (\prod_{s=1}^{t} \alpha(g_s))\chi_i(\prod_{s=1}^{t} g_s).$$

Cayley digraphs of Type-IV, V and VI

Consider the Cayley digraph $Cay(D_{2n}, S)$ with $|S| = 3$, then besides Type-I, Type-II and Type-III studied by Huang (all undirected graphs), it remains to discuss

- Type-IV: $S = \{a^k, ba^{l_1}, ba^{l_2}\}$ ($k \neq \frac{n}{2}$ when n is even);
- Type-V: $S = \{a^{k_1}, a^{k_2}, ba^l\}$ ($k_1 + k_2 \neq n$ *and* $k_1, k_2 \neq \frac{n}{2}$);
- Type-VI: $S = \{a^k, a^{\frac{n}{2}}, ba^l\}$ (only occurs when n is even).

Remark. Cayley digraph of Type-IV is not isomorphic to that of Type-V, and have different H-spectrum.

H-spectrum of Cayley graph $Cay(D_{2n}, S)$ of type IV, V and VI

Lemma (Yu & L, 2018)

The H-spectrum of $Cay(D_{2n}, S)$ *with* $|S| = 3$ *is*

- *Type-IV:*

$$\begin{cases} \{[2]^1, [-2]^1; [\pm\sqrt{a_h(S)}]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\}, & \text{if } n \text{ is odd}, \\ \{[2]^1, [-2]^1, \lambda, -\lambda; [\pm\sqrt{a_h(S)}]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\}, & \text{if } n \text{ is even}, \end{cases}$$

where $\lambda = (-1)^{l_1} + (-1)^{l_2}$ *and*

$$a_h(S) = 2\cos\frac{2\pi h(l_1 - l_2)}{n} - 2\cos\frac{2\pi h(2k)}{n} + 4.$$

H-spectrum of Cayley graph $Cay(D_{2n}, S)$ of type IV, V and VI

- Type-V:

$$\begin{cases} \{[1]^1, [-1]^1; [\pm\sqrt{a_h(S)}]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\} & \textit{if } n \textit{ is odd}, \\ \{[1]^2, [-1]^2; [\pm\sqrt{a_h(S)}]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\} & \textit{if } n \textit{ is even}, \end{cases}$$

where

$$a_h(S) = 4(\sin\frac{2\pi hk_1}{n} + \sin\frac{2\pi hk_2}{n})^2 + 1. \qquad (2)$$

H-spectrum of Cayley graph $Cay(D_{2n}, S)$ of type IV, V and VI

- Type-VI:

$$\{[2]^1, [0]^1, \lambda_1, \lambda_2; [(-1)^h \pm \sqrt{-2\cos\frac{2\pi h(2k)}{n} + 3}]^2 \mid 1 \leq h \leq [\frac{n-1}{2}]\}$$

where $\lambda_1 = (-1)^{\frac{n}{2}} + (-1)^l$ and $\lambda_2 = (-1)^{\frac{n}{2}} - (-1)^l$.

Connectedness of Cayley digraph $Cay(D_{2n}, S)$

Lemma (Yu& L, 2018)

Let $Cay(D_{2n}, S)$ be the Cayley digraph on D_{2n} with respect to S, where $|S| = 3$.

1. *If $Cay(D_{2n}, S)$ is of Type-IV, i.e, $S = \{a^k, ba^{l_1}, ba^{l_2}\}$ $(k \neq \frac{n}{2})$, then $Cay(D_{2n}, S)$ is strongly connected if and only if $(k, l_1 - l_2, n) = 1$.*
2. *If $Cay(D_{2n}, S)$ is of Type-V, i.e, $S = \{a^{k_1}, a^{k_2}, ba^l\}$ $(k_1 + k_2 \neq n$ and $k_1, k_2 \neq \frac{n}{2})$, then $Cay(D_{2n}, S)$ is strongly connected if and only if $(k_1, k_2, n) = 1$.*
3. *If $Cay(D_{2n}, S)$ is of Type-VI, i.e., $S = \{a^k, a^{\frac{n}{2}}, ba^l\}$ (only occurs when n is even), then $Cay(D_{2n}, S)$ is connected if and only if $(\frac{n}{2}, k) = 1$.*

Main results

Theorem (L & Yu, 2018)

Suppose that $Cay(D_{2p}, S)$ *and* $Cay(D_{2p}, T)$ *are two Cayley digraphs of type-IV, where* $S = \{a^k, ba^{l_1}, ba^{l_2}\}$ *and* $T = \{a^{k'}, ba^{l'_1}, ba^{l'_2}\}$, $p \geq 5$ *is an odd prime, and* $l_1 - l_2 \neq \pm 2k(\mod p)$. *Then* $HSpec(Cay(D_{2p}, S))$=$HSpec(Cay(D_{2p}, T))$ *if and only if there exists some* $\mu \in \mathbb{Z}_p^*$ *such that* $l_1 - l_2 = \pm\mu(l'_1 - l'_2)(\mod p), \quad k = \pm\mu k'(\mod p)$.

Proof.

Note that HSpec($Cay(D_{2p}, S)$)=HSpec($Cay(D_{2p}, T)$) if and and only if $\{a_h(S)|1 \leq h \leq \frac{p-1}{2}\} = \{a_h(T)|1 \leq h \leq \frac{p-1}{2}\}$, equivalently,

$$\begin{aligned}&\{2\cos\frac{2\pi hs_1}{p} - 2\cos\frac{2\pi hs_2}{p} \mid 1 \leq h \leq \frac{p-1}{2}\}\\ &= \{2\cos\frac{2\pi ht_1}{p} - 2\cos\frac{2\pi ht_2}{p} \mid 1 \leq h \leq \frac{p-1}{2}\}, \qquad (3)\end{aligned}$$

continued of proof.

where $s_1 \equiv l_2 - l_1 \pmod p$, $s_2 \equiv 2k \pmod p$ and $t_1 \equiv l_2' - l_1' \pmod p$, $t_2 \equiv 2k' \pmod p$. And Equation (3) is equivalent to

$$\begin{aligned} &\{2\cos\frac{2\pi hs_1}{p} - 2\cos\frac{2\pi hs_2}{p} \mid 1 \le h \le p-1\} \\ &= \{2\cos\frac{2\pi ht_1}{p} - 2\cos\frac{2\pi ht_2}{p} \mid 1 \le h \le p-1\}. \end{aligned} \tag{4}$$

$\square$

Main Results

continued of proof.

Since $l_2 \neq l_1 (\mod p)$ and $k \neq \frac{p}{2}$, $s_1 \neq 0 (\mod p)$, $s_2 \neq 0 (\mod p)$. And $s_1 \neq \pm s_2 (\mod p)$ by the assumption.
Choose a $2\cos\frac{2\pi s_1}{p} - 2\cos\frac{2\pi s_2}{p}$ at the left side of Eq. (4), there exists some $1 \leq \mu \leq p-1$ such that

$$2\cos\frac{2\pi s_1}{p} - 2\cos\frac{2\pi s_2}{p} = 2\cos\frac{2\pi \mu t_1}{p} - 2\cos\frac{2\pi \mu t_2}{p}.$$

Let $\omega = e^{\frac{2\pi}{p}i}$. And this is equivalent to

$$\omega^{s_1} + \omega^{-s_1} - \omega^{s_2} - \omega^{-s_2} = \omega^{\mu t_1} + \omega^{-\mu t_1} - \omega^{\mu t_2} - \omega^{-\mu t_2}.$$

$\square$

Main Results

continued of proof.

Since $1 + x + x^2 + \cdots + x^{p-1}$ is the minimal polynomial of ω with respect to the rational field Q, $\omega, \omega^2, \ldots, \omega^{p-1}$ form a basis of $Q(\omega)$. Thus we have

$$s_1 = \pm\mu t_1 \pmod p, \quad s_2 = \pm\mu t_2 \pmod p,$$

equivalently,

$$l_1 - l_2 = \pm\mu(l_1' - l_2') \pmod p, \quad k = \pm\mu k' \pmod p.$$

Conversely, if S and T satisfy the conditions as stated, then it is easy to see that Cay(D_{2p}, S) and Cay(D_{2p}, T) are H-cospectral. □

Main Results

Lemma

Let $X =$Cay(D_{2p}, S) be the Cayley digraph of type-IV with respect to $S = \{a^k, ba^{l_1}, ba^{l_2}\}$. Then $X \cong Cay(D_{2p}, \{a^s, b, ba\})$, for some $s \in \mathbb{Z}_p^*$.

Proof.

Note that $l_1 - l_2 \in \mathbb{Z}_p^*$. It suffices to choose $\sigma_{\lambda,\mu} \in Aut(X)$ with $\lambda = (l_1 - l_2)^{-1}$ and $\mu = -(l_1 - l_2)^{-1} l_2$. Then $\sigma_{\lambda,\mu}(S) = \{a^s, b, ba\}$, where $s = (l_1 - l_2)^{-1} k \in \mathbb{Z}_p$. $\square$

Main Results

Theorem

Let $Cay(D_{2p}, S)$ and $Cay(D_{2p}, T)$ be two Cayley digraphs of type-IV. Then the following statements are equivalent:

(1) *$Cay(D_{2p}, S)$ and $Cay(D_{2p}, T)$ are H-cospectral;*

(2) *$T = S^{\sigma}$, for some $\sigma \in Aut(D_{2p})$;*

(3) *$Cay(D_{2p}, S) \cong Cay(D_{2p}, T)$.*

Proof.

(1)⇒(2). First consider $p \geq 5$. Suppose $S = \{a^k, ba^{l_1}, ba^{l_2}\}$ and $T = \{a^{k'}, ba^{l'_1}, ba^{l'_2}\}$. Then there exist $\sigma_1, \sigma_2 \in Aut(D_{2p})$ such that $S^{\sigma_1} = \{a^s, b, ba\}$ and $T^{\sigma_2} = \{a^t, b, ba\}$, where $s = (l_1 - l_2)^{-1}k, t = (l'_1 - l'_2)^{-1}k' \in Z_p^*$.
Note that $l_1 - l_2 \neq \pm 2k \pmod p$ if and only if $l'_1 - l'_2 \neq \pm 2k' \pmod p$.
If $l_1 - l_2 \neq \pm 2k \pmod p$, then there exist $\mu \in Z_p^*$ such that $l_1 - l_2 = \pm\mu(l'_1 - l'_2)$ and $k = \pm\mu k'$. □

Proof.

Thus $s = (l_1 - l_2)^{-1}k = \pm(l'_1 - l'_2)^{-1}l' = \pm t$ in this case.
If $l_1 - l_2 = \pm 2k(\mod p)$, then $l'_1 - l'_2 = \pm 2k'(\mod p)$, and so

$$2(l_1 - l_2)^{-1}k = \pm 2(l'_1 - l'_2)^{-1}l'(\mod p).$$

Hence we also have $s = \pm t$.
It is easy to check that $\sigma = \sigma_{\lambda,\mu}$ with $\lambda = -1$ and $\mu = 1$ satisfies $T^{\sigma_2} = (S^{\sigma_1})^{\sigma}$, namely, $T = S^{\sigma_2^{-1}\sigma\sigma_1}$ and obviously $\sigma_2^{-1}\sigma\sigma_1 \in Aut(D_{2p})$.
For the case of $p = 3$, it can be verified by a straight calculation.
Clearly, (2)⇒(3) and (3)⇒(1) hold. □

Main Results

Theorem (L & Yu, 2018)

Suppose that $Cay(D_{2p}, S)$ and $Cay(D_{2p}, T)$ are two Cayley digraphs of type-V, where $S = \{a^{k_1}, a^{k_2}, ba^{l}\}$ and $T = \{a^{k'_1}, a^{k'_2}, ba^{l'}\}$, $p \geq 5$ is an odd prime. Then HSpec($Cay(D_{2p}, S)$)=HSpec($Cay(D_{2p}, T)$) if and only if there exists some $\mu \in \mathbb{Z}_p^*$ such that $\{k_1, k_2\} = \mu\{k'_1, k'_2\}$.

Proof.

Note that HSpec($Cay(D_{2p}, S)$)=HSpec($Cay(D_{2p}, T)$) if and and only if $\{a_h(S)|1 \leq h \leq \frac{p-1}{2}\} = \{a_h(T)|1 \leq h \leq \frac{p-1}{2}\}$, equivalently,

$$\begin{aligned} &\{\pm(\sin\frac{2\pi hk_1}{p} + \sin\frac{2\pi hk_2}{p}) \mid 1 \leq h \leq \frac{p-1}{2}\} \\ &= \{\pm(\sin\frac{2\pi hk'_1}{p} + \sin\frac{2\pi hk'_2}{p}) \mid 1 \leq h \leq \frac{p-1}{2}\}, \end{aligned} \tag{5}$$

namely

Main Results

continued of proof.

namely

$$\{\sin\frac{2\pi hk_1}{p}+\sin\frac{2\pi hk_2}{p}\mid 1\le h\le p-1\}$$
$$=\{\sin\frac{2\pi hk_1'}{p}+\sin\frac{2\pi hk_2'}{p}\mid 1\le h\le p-1\}. \tag{7}$$

Choose a $\sin\frac{2\pi k_1}{p}+\sin\frac{2\pi k_2}{p}$ at the left side of Eq. (7), there exists some $1\le\mu\le p-1$ such that

$$\sin\frac{2\pi k_1}{p}+\sin\frac{2\pi k_2}{p}=\sin\frac{2\pi\mu k_1'}{p}+\sin\frac{2\pi\mu k_2'}{p}.$$

Let $\omega=e^{\frac{2\pi}{p}i}$. And this is equivalent to

$$\omega^{k_1}-\omega^{-k_1}+\omega^{k_2}-\omega^{-k_2}=\omega^{\mu k_1'}-\omega^{-\mu k_1'}+\omega^{\mu k_2'}-\omega^{-\mu k_2'}.$$

Main Results

continued of proof.

Note that $k_2 \neq -k_1 (\mod p)$, otherwise $a_h(S) = 0$ for any h, and then HSpec(Cay(D_{2p}, S))=$\{[1]^p, [-1]^p\}$, impossible due to that $Cay(D_{2p}, S)$ is not disjoint union of edges. In addition, $k_i \neq -k_i (\mod p)$ for $i = 1, 2$, and $k_1 \neq k_2 (\mod p)$.
By the same way as previously, $\omega^{k_1}, \omega^{-k_1}, \omega^{k_2}, \omega^{-k_2}$ are linearly independent over $Q(\omega)$. Thus, $k_1 = \mu k_1'$ or $k_1 = \mu k_2'$ and $k_2 = \mu k_1'$ or $k_1 = \mu k_2'$, and so

$$\{k_1, k_2\} = \mu\{k_1', k_2'\}.$$

Conversely, if S and T satisfy the condition as stated, then clearly $Cay(D_{2p}, S)$ and $Cay(D_{2p}, T)$ are H-cospectral.

$\square$

Main Results

Lemma

Let $X =$ Cay(D_{2p}, S) be a Cayley digraph of type-V with $S = \{a^{k_1}, a^{k_2}, ba^l\}$. Then X and Cay$(D_{2p}, \{a^s, b, a\})$ are Cayley isomorphic for some $s \in \mathbb{Z}_p^*$.

Proof.

Note that $k_1 \in \mathbb{Z}_p^*$. It suffices to choose $\sigma_{\lambda,\mu} \in Aut(D_{2p})$ with $\lambda = k_1^{-1}$ and $\mu = -k_1^{-1}l$. Then $\sigma_{\lambda,\mu}(S) = \{a^s, b, a\}$, where $s = k_1^{-1}k_2 \in \mathbb{Z}_p^*$. □

Main Results

Lemma

Suppose $S = \{a^s, b, a\}$ *and* $T = \{a^t, b, a\}$*, where* $s, t \in \mathbb{Z}_p^*$*. Then the following statements are equivalent:*

(1) $Cay(D_{2p}, S)$ *and* $Cay(D_{2p}, T)$ *are H-cospectral;*

(2) $s = t \pmod p$ *or* $st = 1 \pmod p$*;*

(3) $T = S^\sigma$*, for some* $\sigma \in Aut(D_{2p})$*.*

Proof.

(1) $\Leftrightarrow$ (2) follows immediately from the previous Theorem. (2) $\Rightarrow$ (3) follows by taking $\sigma = \sigma_{\lambda,\mu}$ with $\lambda = t$ and $\mu = 0$ when $st = 1 \pmod p$. Obviously (3) $\Rightarrow$ (1) holds. □

Main Results

Theorem

Suppose $Cay(D_{2p}, S)$ and $Cay(D_{2p}, T)$ are two Cayley digraphs of type-V. Then the following are equivalent:

- $Cay(D_{2p}, S)$ *and* $Cay(D_{2p}, T)$ *are H-cospectral;*
- $T = S^{\sigma}$, *for some* $\sigma \in Aut(D_{2p})$;
- $Cay(D_{2p}, S) \cong Cay(D_{2p}, T)$.

Thanks for your attention!