On equitable partitions of divisible design graphs

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This is joint work with S. Goryainov and L. Shalaginov

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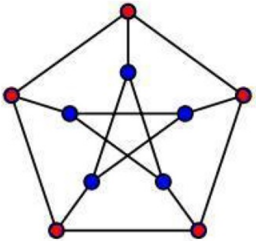
Equitable t-partition of a graph Γ

- $V(\Gamma) = V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_t$ (partition of the vertex set of Γ),
- every vertex of V_i has exactly p_{ij} neighbours of V_j,
- $P := (p_{ij})_{t \times t}$ — the quotient matrix of equitable partition.



$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Previous results

- Equitable 2-partitions of a hypercube (D. G. Fon-Der-Flaas, 2007 ([1]))
- Equitable 2-partitions of Johnson graphs J(v, 3) with eigenvalue λ_2 for odd v (A. L. Gavrilyuk, S. V. Goryainov, 2013 ([2]))
- Equitable t-partitions of Latin-square graphs whose quotient matrix does not have an eigenvalue -3 (R. A. Bailey, P. J. Cameron, A. L. Gavrilyuk, S. V. Goryainov, 2018 ([3]))
- Equitable 2-partitions of Hamming graphs $H(n, q)$ with eigenvalue λ_2 (I. Mogilnykh, A. Valyuzhenich, 2018)

[1] D. G. Fon-Der-Flaass, Perfect 2-colorings of a hypercube, *Siberian Math. J.* **48** (2007), P. 740–745.
[2] A. L. Gavrilyuk, S. V. Goryainov, On perfect 2-colorings of Johnson graphs $J(v, 3)$, *J. Comb. Des.* **21** (2013), P. 232–252.
[3] R. A. Bailey, P. J. Cameron, A. L. Gavrilyuk, S. V. Goryainov, Equitable partitions of Latin-square graphs. https://arxiv.org/abs/1802.01001

Eigenvalues of quotient matrices

Let P be the quotient matrix of a t-partition of a graph Γ. Then each eigenvalue of P is an eigenvalue of the adjacency matrix of Γ ([4]).

If $t = 2$, then exactly one non-principal eigenvalue θ of Γ is an eigenvalue of P.

[4] C. Godsil, G. F. Royle, Algebraic Graph Theory. *Springer, New York*. 2001

Merging of equitable 2-partitions

Lemma 1 ([5])

Let $\Gamma = (V, E)$ be a graph. Let $V_1'\dot\cup V_2'$, $V_1''\dot\cup V_2''$ be two equitable 2-partitions of Γ corresponding to the same eigenvalue θ. If $V_1' \cap V_1'' = \emptyset$ holds, then the partition

$$(V_1' \cup V_1'')\dot\cup(V \setminus (V_1' \cup V_1''))$$

is a perfect 2-colouring corresponding to the eigenvalue θ.

[5] D. S. Krotov, On Perfect Colorings of the Halved 24-Cube. http://arxiv.org/abs/0803.0068

Divisible design graphs

A k-regular graph Γ with v vertices is called a divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if the vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly λ_1 common neighbors, and two vertices from different classes have exactly λ_2 common neighbors.

A divisible design graph has at most five eigenvalues, which can be expressed in terms of the parameters ([6]).

[6] W. H. Haemers, H. Kharaghani, M. A. Meulenberg, Divisible design graphs. *J. of Comb. Theory*. **118** (2011) P. 978–992

Construction based on Hadamard matrix

Consider this two Hadamard matrices:

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

Replace each entry with value:

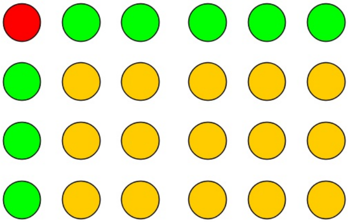
- -1 by $J_n - I_n$,
- $+1$ by I_n.

By doing this, we obtain two series of divisible design graphs ([6]). For the first matrix, the DDG is the $(4 \times n)$-lattice. And we denote graphs obtained from the second matrix as $HL(n)$.

[6] W. H. Haemers, H. Kharaghani, M. A. Meulenberg, Divisible design graphs. *J. of Comb. Theory*. **118** (2011) P. 978–992

$(4 \times n)$-lattice

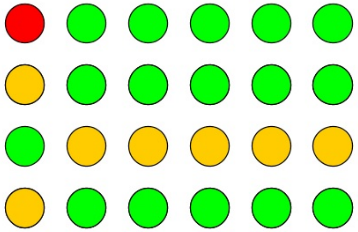
- $(4 \times n)$-lattice is a divisible design graph with parameters $(4n, n+2, n-2, 2, 4, n)$.
- $(4 \times n)$-lattice has four eigenvalues: $\pm 2, n-2, n+2$.



$$(i,j) \sim \{(i,k'), (k'',j),\ \forall k' \neq j, k'' \neq i\}$$

$HL(n)$ graph

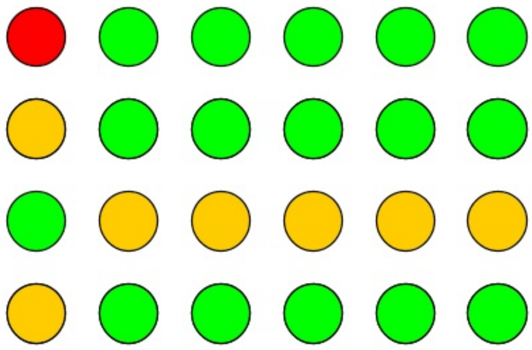
- $HL(n)$ is a divisible design graph with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$.
- $HL(n)$ has five eigenvalues: $\pm 2, \pm(n-2), n+2$.



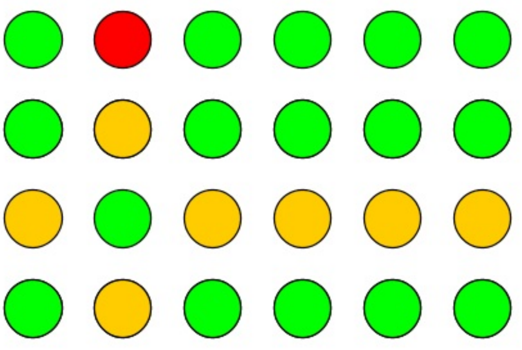
$$(i,j) \sim \{(i,k),(i+1,k),(i-1,k)\ \forall k \neq j\} \cup \{(i+2,j)\}$$

We will call row i and row j ($i \neq j$) neighboring rows if $i - j \equiv \pm 1 \pmod 4$ and opposite rows otherwise.

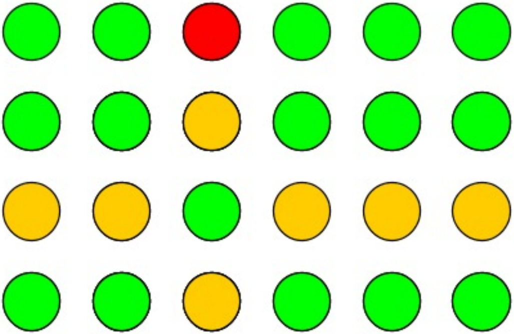
$HL(n)$ graph



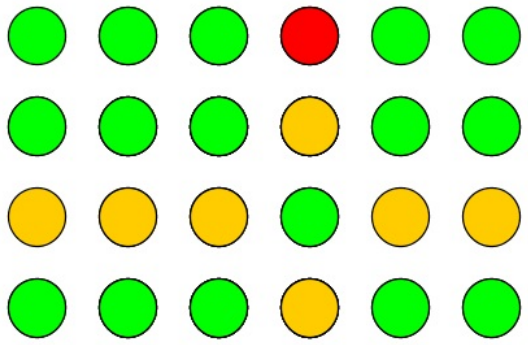
$HL(n)$ graph



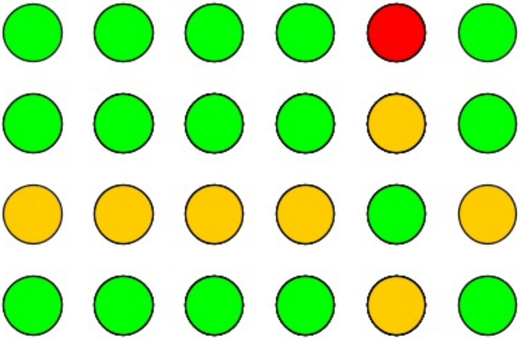
$HL(n)$ graph



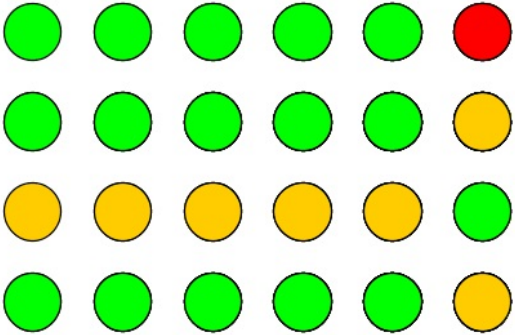
$HL(n)$ graph



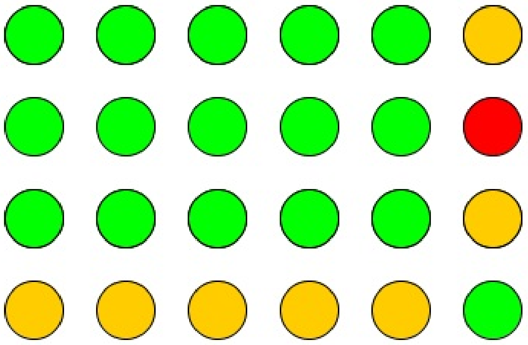
$HL(n)$ graph



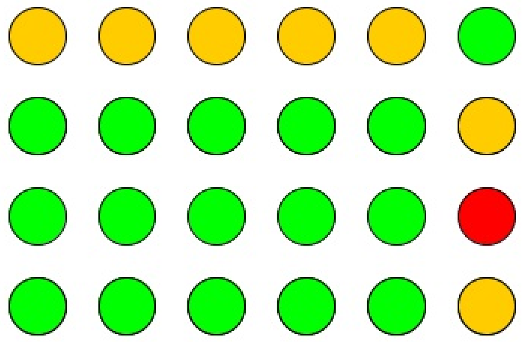
$HL(n)$ graph



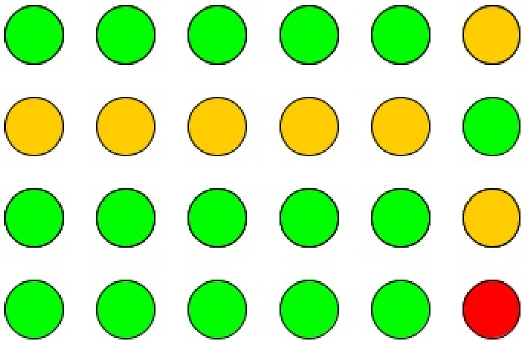
$HL(n)$ graph



$HL(n)$ graph



$HL(n)$ graph



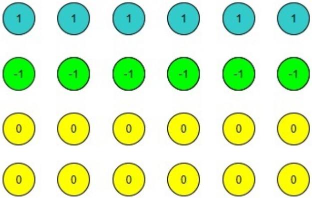
Eigenfunction of a graph

Let Γ be a graph with the adjacency matrix A, and θ be its eigenvalue. A function $f : V(\Gamma) \longrightarrow \mathbb{R}$ is called an eigenfunction of the graph Γ corresponding to the eigenvalue θ if $f \neq 0$ and

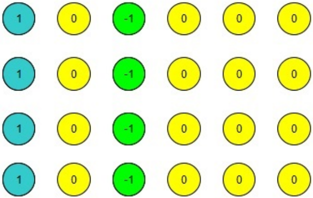
$$\theta \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds for any its vertex x, where $N(x)$ is the neighborhood of x.

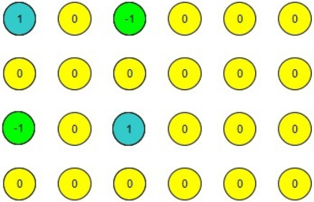
Eigenfunctions of $(4 \times n)$-lattice



$\theta = n - 2$



$\theta = 2$



$\theta = -2$

Correspondence between equitable 2-partitions and eigenfunctions

Let $V_1 \dot\cup V_2$ be a partition of the vertex set of a regular graph Γ. For any $a, b \in \mathbb{R}$, consider a function $V(\Gamma) \longrightarrow \mathbb{R}$ such that for any $z \in V(\Gamma)$

$$f(z) = \begin{cases} a; if \ z \in V_1; \\ b; if \ z \in V_2. \end{cases}$$

Let θ be a non-principal eigenvalue of the graph Γ.

Correspondence between equitable 2-partitions and eigenfunctions

Lemma 2

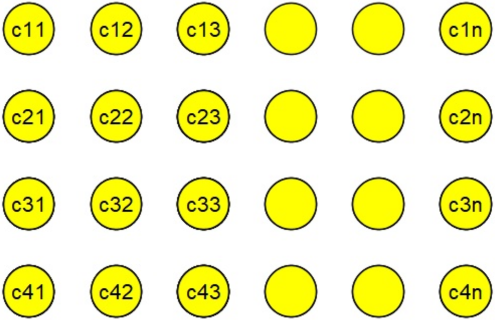
The following statements are equivalent.

(1) The partition $V_1 \dot{\cup} V_2$ is an equitable 2-partition of the graph Γ corresponding to the eigenvalue θ, and $\begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector of the quotient matrix P corresponding to the eigenvalue θ;

(2) The function f is an eigenfunction of Γ corresponding to the eigenvalue θ.

Correspondence between equitable 2-partitions and eigenfunctions

Let $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ be the quotient matrix of a equitable 2-partition $V_1 \dot{\cup} V_2$ of $(4 \times n)$-lattice such that $\theta_1 = n - 2$ is an eigenvalue of P.



$$c_{ij} \in \{0, 1\} \; \forall i \in \{1, 2, 3, 4\}, j \in \{1, \ldots, n\}$$

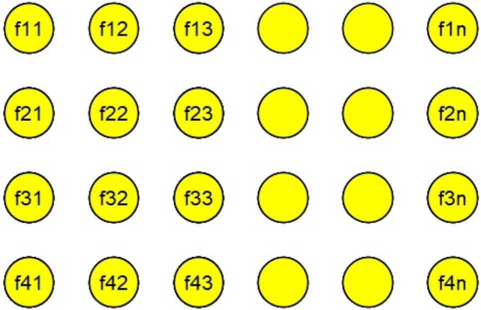
Correspondence between equitable 2-partitions and eigenfunctions

This implies that the function $f : V(\Gamma) \longrightarrow \mathbb{R}$ defined by the rule

$$f(z) = \begin{cases} a; if \ z \in V_1; \\ b; if \ z \in V_2. \end{cases}$$

is an θ_1-eigenfunction of $(4 \times n)$-lattice, where $\begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector of P corresponding to the eigenvalue $\theta_1 = n - 2$. Note that $a \neq b$ holds.

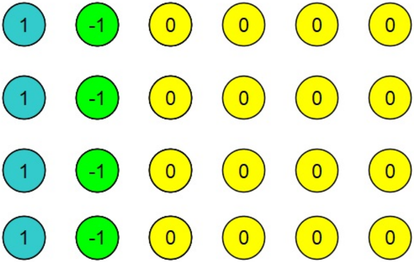
Correspondence between equitable 2-partitions and eigenfunctions



$f_{ij} \in \{a, b\}$ are values of the function f

Correspondence between equitable 2-partitions and eigenfunctions

Let us consider the θ_2-eigenfunction, where $\theta_2 = 2$:



Since the θ_1-eigenfunction f and any θ_2-eigenfunction are orthogonal, their scalar product equals 0. Thus, we obtain

$$f_{11} - f_{12} + f_{21} - f_{22} + f_{31} - f_{32} + f_{41} - f_{42} = 0.$$

Correspondence between equitable 2-partitions and eigenfunctions

$$f_{11} + f_{21} + f_{31} + f_{41} = f_{12} + f_{22} + f_{32} + f_{42}$$

Let the number a appear x times in the sequence $f_{11}, f_{21}, f_{31}, f_{41}$ and y times in the sequence $f_{12}, f_{22}, f_{32}, f_{42}$, where $x, y \in \{0, 1, 2, 3, 4\}$. Then we have

$$xa + (4 - x)b = ya + (4 - y)b,$$

$$xa - xb = ya - yb,$$

$$(x - y)(a - b) = 0.$$

Since $a \neq b$ holds, we have $x = y$. This means that the sequences $c_{11}, c_{21}, c_{31}, c_{41}$ and $c_{12}, c_{22}, c_{32}, c_{42}$ have the same number of 1s. Thus, we have

$$c_{11} + c_{21} + c_{31} + c_{41} = c_{12} + c_{22} + c_{32} + c_{42}.$$

Correspondence between equitable 2-partitions and eigenfunctions

Using this approach with other eigenfunctions for $\theta_2 = \pm 2$ we obtain the system of equalities.
This system gives strong necessary conditions for our equitable 2-partition $V_1 \dot\cup V_2$.

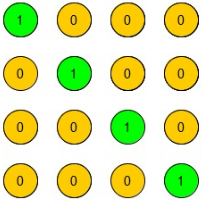
By doing the same for other two eigenvalues we can find all different equitable 2-partitions for $(4 \times n)$-lattice.

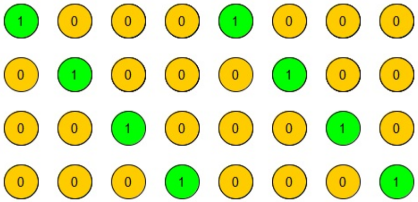
Equitable 2-partitions of $(4 \times n)$-lattice

Theorem 1

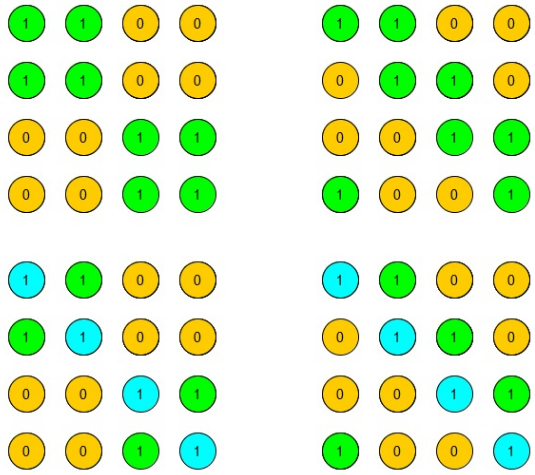
$(4 \times n)$-lattice has only following equitable 2-partitions:
(1) For $\theta = n - 2$ each row lies entirely in V_1 or V_2.
(2) For $\theta = 2$ each column lies entirely in V_1 or V_2.
(3) For $\theta = -2$ exactly t vertices from each column and exactly m vertices from each row lie in V_1 and:

- if $n \equiv 0 \pmod 4$, $t = 1, 2, 3$ and $m = \frac{n}{4}, \frac{n}{2}, \frac{3n}{4}$;
- if $n \equiv 2 \pmod 4$, $t = 2$ and $m = \frac{n}{2}$;
- if $n \equiv 1, 3 \pmod 4$, equitable 2-partitions for $\theta = -2$ do not exist.

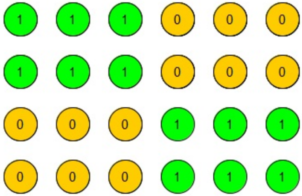

1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1

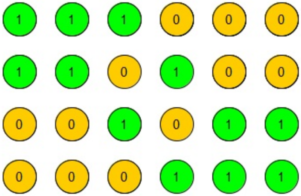

1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0
0 0 1 0 0 0 1 0
0 0 0 1 0 0 0 1

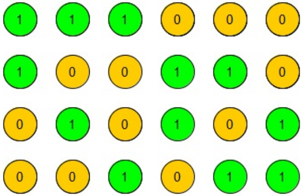
Equitable 2-partition of $(4 \times n)$-lattice for $\theta = -2$



Equitable 2-partition of $(4 \times n)$-lattice for $\theta = -2$







Equitable 2-partition of $(4 \times n)$-lattice for $\theta = -2$

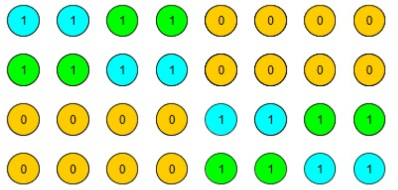
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0
0	0	0	0	1	1	1	1
0	0	0	0	1	1	1	1

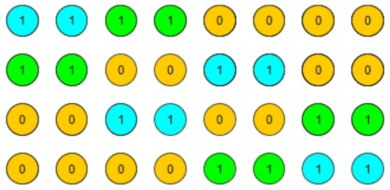
1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1
0	0	0	0	1	1	1	1

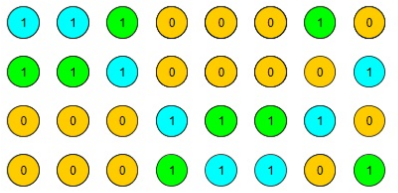
1	1	1	0	0	0	1	0
1	1	1	0	0	0	0	1
0	0	0	1	1	1	1	0
0	0	0	1	1	1	0	1

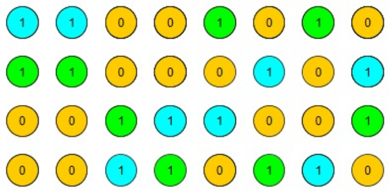
1	1	0	0	1	0	1	0
1	1	0	0	0	1	0	1
0	0	1	1	1	0	0	1
0	0	1	1	0	1	1	0

Equitable 2-partition of $(4 \times n)$-lattice for $\theta = -2$

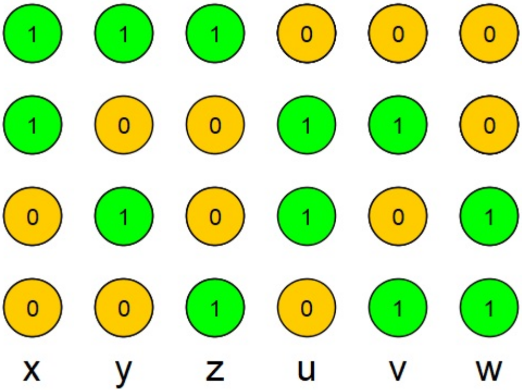








Equitable 2-partition of $(4 \times n)$-lattice for $\theta = -2$



Equitable 2-partition of $(4 \times n)$-lattice for $\theta = -2$

$$\begin{cases} x + y + z = \frac{n}{2} \\ x + u + v = \frac{n}{2} \\ y + u + w = \frac{n}{2} \\ z + v + w = \frac{n}{2} \end{cases} \qquad \begin{cases} x + y + z = \frac{n}{2} \\ x = w \\ y = v \\ z = u \end{cases}$$

Equitable 2-partitions of $(m \times n)$-lattice

Theorem 2

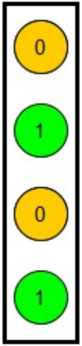
$(m \times n)$-lattice has only following equitable 2-partitions:
(1) For $\theta = n - 2$ each row lies entirely in V_1 or V_2.
(2) For $\theta = m - 2$ each column lies entirely in V_1 or V_2.
(3) For $\theta = -2$ exactly t vertices from each column and exactly m vertices from each row lie in V_1.

Equitable 2-partitions of $HL(n)$

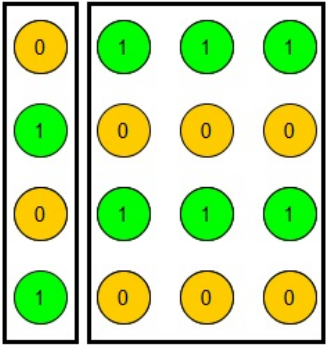
Theorem 3

$HL(n)$ lattice has only following equitable 2-partitions:
(1) For $\theta = n-2$ two neighbouring rows lie in V_1 and two other rows lies in V_2.
(2) For $\theta = -(n-2)$ two opposite rows lie in V_1 and two other rows lies in V_2.
(3) ...

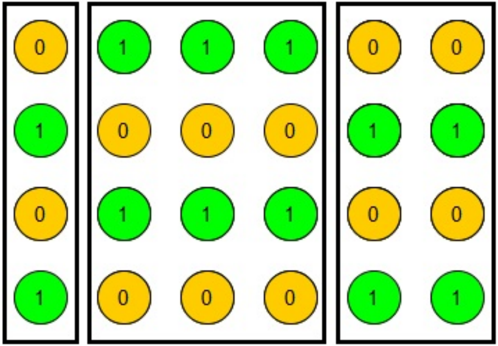
Equitable 2-partition of $HL(n)$ for $\theta = 2$



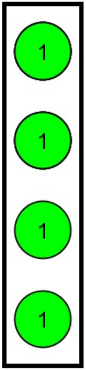
Equitable 2-partition of $HL(n)$ for $\theta = 2$



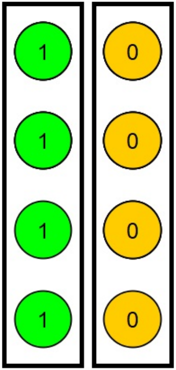
Equitable 2-partition of $HL(n)$ for $\theta = 2$



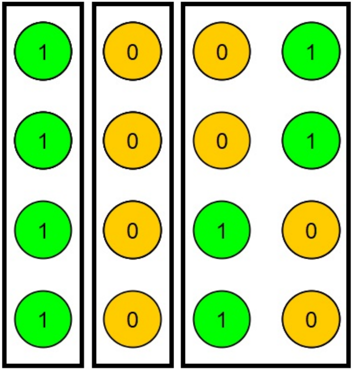
Equitable 2-partition of $HL(n)$ for $\theta = -2$



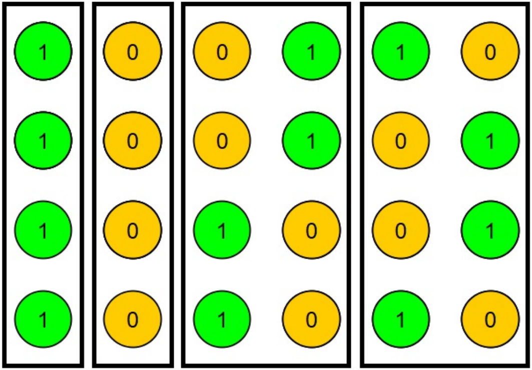
Equitable 2-partition of $HL(n)$ for $\theta = -2$



Equitable 2-partition of $HL(n)$ for $\theta = -2$



Equitable 2-partition of $HL(n)$ for $\theta = -2$



Open questions

- Finding equitable t-partitions of $(m \times n)$-lattice for any t.
- Finding equitable partitions of divisible design graphs obtained from other regular graphical Hadamard matrices.
- Finding equitable partitions of other divisible design graphs.

Thanks for your attention!