On eigenfunctions of Hamming graphs with minimum support

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Let $G = (V, E)$ be a simple graph.

Definition

A real–valued function $f : V \longrightarrow \mathbb{R}$ is called a *λ–eigenfunction* of G if the equality

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds for any $x \in V$.

The set of all λ–eigenfunctions of G is called a *λ–eigenspace* of G.

Example of an eigenfunction

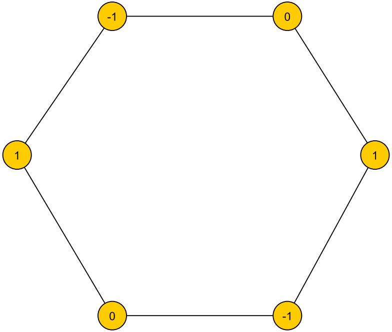

Figure: Example of a (-1)–eigenfunction.

Basic definitions

Definition

The support of a real–valued function f is the set of nonzeros of f.

The cardinality of the support of f is denoted by $|f|$.

Basic definitions

$\Sigma_q = \{0, 1, \ldots, q-1\}$.

Definition

The Hamming distance $d(x, y)$ between vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ from Σ_q^n is the number of positions i such that $x_i \neq y_i$.

Definition

The Hamming graph $H(n, q)$ is a graph whose vertex set is Σ_q^n and two vertices are adjacent if the Hamming distance between them equals 1.

Basic definitions

It is well known that the set of eigenvalues of $H(n,q)$ is $\{\lambda_i(n,q) = n(q-1) - q \cdot i \mid i = 0, 1, \ldots, n\}$.

Denote by $U_i(n,q)$ the $\lambda_i(n,q)$–eigenspace of $H(n,q)$. The direct sum of subspaces

$$U_i(n,q) \oplus U_{i+1}(n,q) \oplus \ldots \oplus U_j(n,q)$$

for $0 \leq i \leq j \leq n$ is denoted by $U_{[i,j]}(n,q)$.

Problem

To find the minimum cardinality of the support of eigenfunctions of the distance–regular graphs.

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This problem is directly related to the problem of finding the minimum cardinality of the bitrades.

Connection with bitrades

Definition

A 1-perfect bitrade is a pair (T_0, T_1) of disjoint nonempty sets of vertices of $H(n, q)$ such that for every ball B of radius 1 the equality $|B \cap T_0| = |B \cap T_1| = a$ holds, where $a \in \{0, 1\}$.

Definition

The size of a bitrade (T_0, T_1) is $|T_0| + |T_1|$.

Example of bitrade

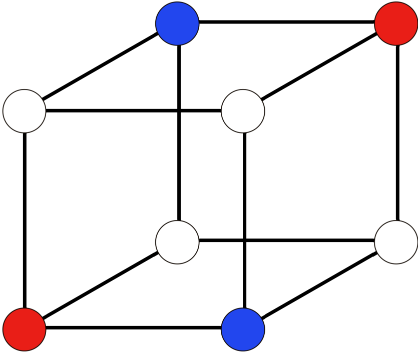

Figure: Bitrade in $H(3,2)$.

T_0 are red vertices, T_1 are blue vertices.
This bitrade has the size 4.

Example of bitrades

Definition

A set C of vertices of a regular graph G is called a 1-perfect code iff every ball of radius 1 contains exactly one element of C.

Example. Let C_1 and C_2 be 1–perfect codes in $H(n, q)$. Then the pair $(C_1 \setminus C_2, C_2 \setminus C_1)$ is a 1–perfect bitrade.

Bitrades and eigenfunctions

Let (T_0, T_1) be a 1-perfect bitrade in $H(n, q)$.
We define the function $\chi(T_0, T_1) : \Sigma_q^n \longrightarrow \{-1, 0, 1\}$ by the following rule:

$$\chi(T_0, T_1)(x) = \begin{cases} 1, & \text{if } x \in T_0; \\ -1, & \text{if } x \in T_1; \\ 0, & \text{otherwise.} \end{cases}$$

We note that $\chi(T_0, T_1)$ is a (-1)–eigenfunction of $H(n, q)$.

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We note that $\chi(T_0, T_1)$ is a (-1)–eigenfunction of $H(n,q)$.

Thus, if we know the minimum size of the support of a (-1)–eigenfunction of $H(n,q)$, we obtain the lower bound for the size of a 1-perfect bitrade in $H(n,q)$.

The Johnson graphs $J(n, w)$

Definition

The Johnson graph $J(n, w)$ is a graph whose vertices are the binary vectors of length n with w ones, and two vertices are adjacent if they have exactly $w - 1$ common ones.

Let $'$ be a bijection of two disjoint subsets M and M' of coordinate positions of size i. For a subset I of M denote the set of its images by I': $I' = \{m' : m \in I\} \subseteq M'$. Define the function $f^{i,w,n}$ on the vectors of weight w and length n as follows:

$$f^{i,w,n}(x) = (-1)^{|M \cap supp(x)|}, \text{if } |supp(x) \cap (M \cup M')| = i \text{ and}$$

$$(supp(x) \cap M)' \cup (supp(x) \cap M') = M',$$

and $f^{i,w,n}(x) = 0$ otherwise.

Theorem (Vorob'ev, Mogilnykh, V., 2018)

Let i, w be positive integers, $w \geq i$. There is $n_0(i, w)$ such that for all $n \geq n_0(i, w)$ and any nonzero $\lambda_i(n, w)$-eigenfunction f of $J(n, w)$ the following holds:

$$|f| \geq 2^i \binom{n-2i}{w-i},$$

with equality attained only for the function $f^{i,w,n}$ up to multiplication by a scalar.

Some results

The problem of finding the minimum size of the support of eigenfunctions was studied

- for the Doob graphs (Bespalov, 2018)
- for the bilinear forms graphs (Sotnikova, 2018)
- for the cubic distance-regular graphs (Sotnikova, 2018)
- for the Paley graphs (Goryainov, Kabanov, Shalaginov, V., 2018)

Basic definitions

- The Cartesian product $G \square H$ of graphs G and H is a graph with the vertex set $V(G) \times V(H)$; and any two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G.
- Let $G = G_1 \square G_2$, $f_1 : V(G_1) \longrightarrow \mathbb{R}$ and $f_2 : V(G_2) \longrightarrow \mathbb{R}$. We define the product $f_1 \cdot f_2$ by the following rule:
  $(f_1 \cdot f_2)(x, y) = f_1(x) f_2(y)$ for $(x, y) \in V(G)$.

Lemma 1

Let G_1 and G_2 be two graphs, λ and μ be eigenvalues of G_1 and G_2 respectively, x and y be eigenvectors for λ and μ. Then the graph $G_1 \square G_2$ has the eigenvalue $\lambda + \mu$ and $x \otimes y$ is the eigenvector corresponding to $\lambda + \mu$.

Since

$$\lambda_i(m, q) + \lambda_j(n, q) = \lambda_{i+j}(m + n, q)$$

and $H(m + n, q) = H(m, q) \square H(n, q)$, by Lemma 1 we obtain the following result:

Lemma 2

Let $f_1 \in U_i(m, q)$ and $f_2 \in U_j(n, q)$. Then $f_1 \cdot f_2 \in U_{i+j}(m + n, q)$.

We define the function $a_1(k,m) : \Sigma_q^2 \longrightarrow \mathbb{R}$ for $k, m \in \Sigma_q$ by the following rule:

$$a_1(k,m)(x,y) = \begin{cases} 1, & \text{if } x = k \text{ and } y \neq m; \\ -1, & \text{if } y = m \text{ and } x \neq k; \\ 0, & \text{otherwise.} \end{cases}$$

We note that $|a_1(k,m)| = 2(q-1)$ for $k, m \in \Sigma_q$. The set of functions $a_1(k,m)$ where $k, m \in \Sigma_q$ is denoted by A_1.

Example

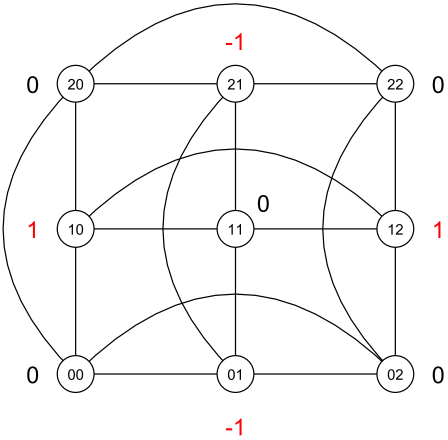

Figure: Function $a_1(1, 1)$ for $q = 3$.

We define the function $a_2(k,m) : \Sigma_q \longrightarrow \mathbb{R}$ for $k, m \in \Sigma_q$ and $k \neq m$ by the rule:

$$a_2(k,m)(x) = \begin{cases} 1, & \text{if } x = k; \\ -1, & \text{if } x = m; \\ 0, & \text{otherwise.} \end{cases}$$

The set of functions $a_2(k,m)$ where $k, m \in \Sigma_q$ and $k \neq m$ is denoted by A_2.

Example

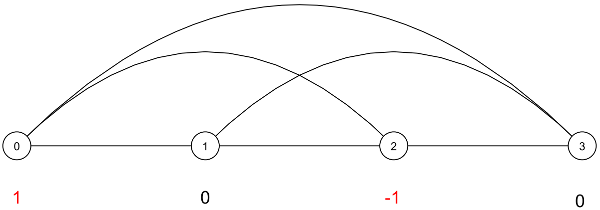

Figure: Function $a_2(0,2)$ for $q=4$.

Basic definitions

Let $A_3 = \{f : \Sigma_q \longrightarrow \mathbb{R} \mid f \equiv 1\}$.

We define the function $a_4(m) : \Sigma_q \longrightarrow \mathbb{R}$ for $m \in \Sigma_q$ by the rule:

$$a_4(m)(x) = \begin{cases} 1, & \text{if } x = m; \\ 0, & \text{otherwise.} \end{cases}$$

The set of functions $a_4(m)$ where $m \in \Sigma_q$ is denoted by A_4.

Class $F_1(n, q, i, j)$

Let $n \geq i + j$. We say that a function $f : \Sigma_q^n \longrightarrow \mathbb{R}$ belongs to the class $F_1(n, q, i, j)$ if

$$f = c \cdot \prod_{k=1}^{i} g_k \cdot \prod_{k=1}^{n-i-j} h_k \cdot \prod_{k=1}^{j-i} v_k,$$

where c is a constant, $g_k \in A_1$ for $k \in [1, i]$, $h_k \in A_3$ for $k \in [1, n - i - j]$ and $v_k \in A_4$ for $k \in [1, j - i]$.

Since $A_1 \subset U_1(2, q)$, $A_3 \subset U_0(1, q)$ and $A_4 \subset U_{[0,1]}(1, q)$, by Lemma 2 we have that $f \in U_{[i,j]}(n, q)$.

Class $F_2(n,q,i,j)$

Let $i+j>n$. We say that a function $f:\Sigma_q^n \longrightarrow \mathbb{R}$ belongs to the class $F_2(n,q,i,j)$ if

$$f = c \cdot \prod_{k=1}^{n-j} g_k \cdot \prod_{k=1}^{i+j-n} h_k \cdot \prod_{k=1}^{j-i} v_k,$$

where c is a constant, $g_k \in A_1$ for $k \in [1, n-j]$, $h_k \in A_2$ for $k \in [1, i+j-n]$ and $v_k \in A_4$ for $k \in [1, j-i]$.

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Let $f(x_1, x_2, \ldots, x_n)$ be a function and $\sigma \in S_n$. Let $f_\sigma(x) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$.

Theorem (V., 2017)

Let $f : \Sigma_q^n \longrightarrow \mathbb{R}$, f be a $\lambda_1(n,q)$–eigenfunction of $H(n,q)$, $q \geq 3$ and $f \not\equiv 0$. Then $|f| \geq 2(q-1)q^{n-2}$. Moreover, the equality $|f| = 2(q-1)q^{n-2}$ holds if and only if $f_\sigma \in F_1(n,q,1,1)$ for some permutation $\sigma \in S_n$.

Previous results for the Hamming graphs

Theorem 1 (V., Vorob'ev, 2017)

Let $f : \Sigma_q^n \longrightarrow \mathbb{R}$, $f \in U_{[i,j]}(n, q)$, $n \geq i + j$, $q \geq 3$ and $f \not\equiv 0$. Then $|f| \geq 2^i(q-1)^i q^{n-i-j}$. Moreover, the equality $|f| = 2^i(q-1)^i q^{n-i-j}$ holds if and only if $f_\sigma \in F_1(n, q, i, j)$ for some permutation $\sigma \in S_n$.

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Theorem 1 implies that functions from $U_{[i,j]}(n,q)$ with the minimum size of the support have two interesting properties:

- such functions take only 3 distinct values
- such functions are equal to a tensor product of several elementary eigenfunctions of the Hamming graphs of dimensions not greater that 2.

Applying Theorem 1 for $i = j$ we obtain the following result:

Corollary (V., Vorob'ev, 2017)

Let $f : \Sigma_q^n \longrightarrow \mathbb{R}$, $f \in U_i(n,q)$, $i \leq \lfloor \frac{n}{2} \rfloor$, $q \geq 3$ and $f \not\equiv 0$. Then $|f| \geq 2^i(q-1)^i q^{n-2i}$. Moreover, the equality $|f| = 2^i(q-1)^i q^{n-2i}$ holds if and only if $f_\sigma \in F_1(n,q,i,i)$ for some permutation $\sigma \in S_n$.

Theorem 2 (V., Vorob'ev, 2018)

Let $f : \Sigma_q^n \longrightarrow \mathbb{R}$, $f \in U_{[i,j]}(n,q)$, $i + j > n$, $q \geq 4$ and $f \not\equiv 0$. Then $|f| \geq 2^i(q-1)^{n-j}$. Moreover, for $i = j$ and $q \geq 5$ the equality $|f| = 2^i(q-1)^{n-i}$ holds if and only if $f_\sigma \in F_2(n,q,i,i)$ for some permutation $\sigma \in S_n$.

Main theorem

Corollary (V., Vorob'ev, 2018)

Let $f : \Sigma_q^n \longrightarrow \mathbb{R}$, $f \in U_i(n, q)$, $i > \lfloor \frac{n}{2} \rfloor$, $q \geq 4$ and $f \not\equiv 0$. Then $|f| \geq 2^i(q-1)^{n-i}$. Moreover, for $q \geq 5$ the equality $|f| = 2^i(q-1)^{n-i}$ holds if and only if $f_\sigma \in F_2(n, q, i, i)$ for some permutation $\sigma \in S_n$.

The main idea of the proof

Let $f : \Sigma_q^n \longrightarrow \mathbb{R}$. We define the function $f_k : \Sigma_q^{n-1} \longrightarrow \mathbb{R}$ by the rule $f_k(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{n-1}, k)$.

Lemma

Let $f : \Sigma_q^n \longrightarrow \mathbb{R}$ and $f \in U_{[i,j]}(n, q)$. Then the following statements are true:

1. $f_k - f_m \in U_{[i-1,j-1]}(n-1, q)$ for $k, m \in \Sigma_q$.
2. $\sum_{k=0}^{q-1} f_k \in U_{[i,j]}(n-1, q)$.
3. $f_k \in U_{[i-1,j]}(n-1, q)$ for $k \in \Sigma_q$.

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Thank you for your attention!