Trades in Combinatorial Configurations

Denis Krotov

Sobolev Institute of Mathematics
Novosibirsk

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Trades (introduction)

- Trades reflect possible difference between two combinatorial objects of the same type (Steiner systems or other designs, subspace designs, latin squares or hypercubes, perfect codes, MDS codes, MRD (maximum rank distance) codes, orthogonal arrays, ...)
- If C' and C'' are two objects with the same parameters (e.g., Steiner triple systems STS(13)), then the pair $(C' \backslash C'', C'' \backslash C')$ is a bitrade and $C' \backslash C''$, $C'' \backslash C'$ are trades.
- However, trades can be defines independently from the "complete" objects of given type, they are not necessarily embedded in such objects, and they can exist for parameters for which complete objects do not exist.

The plan

1. Some general facts [1]:
   - We will consider a rather general class of trades that generalizes several well-known types of trades, including latin trades, Steiner $(k-1,k,v)$ trades and their subspace analogs, extended 1-perfect trades.
   - A characterization of the trades in terms of induced subgraphs and in terms of eigenfunctions of the graph.
   - A characterization of the minimum trades in terms of isometrical distance-regular subgraphs.

2. Partial examples: latin hypercubes, Steiner systems, q-ary (subspace) Steiner systems, extended 1-perfect codes, transversales of latin squares.

[1] D.Krotov, I.Mogilnykh, V.Potapov. To the theory of q-ary Steiner and other-type trades, *Discr. Math.* 339(3) 1150–1157, 2016

Definition: eigenfunction, eigenvalue

An eigenfunction og a graph $\Gamma = (V, E)$ is a function $f : V \to \mathbb{R}$, $f \not\equiv 0$, satisfying

$$\sum_{y \in \Gamma_1(x)} f(y) = \theta f(x) \qquad (1)$$

for every x from V and some constant θ, which is called an eigenvalue of the graph Γ.

Definition: distance-regular graph

A connected regular graph is called distance regular iff the bipartite subgraph between any two cocentered spheres is biregular.

Clique systems, (r, s) pairs

- Let Γ be a connected regular graph of degree r. Assume that there exists a set S of $s+1$-cliques of Γ such that every edge is included is exactly m cliques from S. Then (Γ, S) is an (r, s) pair.
- If Γ is a distance regular graph and $-r/s$ is its smallest eigenvalue, then (r, s) is called a Delsarte pair.

Definition: clique design

- Assume that (Γ, S) is an (r, s) pair. Define an S-design, or clique design, as a set of vertices of Γ that intersect with every clique from S in exactly one vertex.
- Examples of clique designs in distance-regular graphs:
  latin squares and latin hypercubes (MDS codes with distance 2) (in Hamming graphs),
  Steiner systems $\mathrm{S}(k-1, k, v)$ (in Johnson graphs),
  their subspace analogs (in Grassmann graphs),
  extended 1-perfect binary codes (in halved hypercubes),
  MRD codes (in bilinear form graphs),
  transversales of latin squares (in latin-square graphs).

Definition: bitrades

- Let S be an (r,s) pair. A pair (T_0, T_1) of disjoint nonempty vertex sets of Γ is called an S-bitrade, or clique bitrade, if every clique from S either intersects with each of T_0, T_1 in one vertex, or disjoint with both T_0, T_1.
- Then, T_0 and T_1 are called trades (sometimes, trade mates, or legs).
- So, "bitrade = (trade, trade)"
  an alternative terminology:
  "trade = (leg, leg)".

Bitrade criterion

Theorem (DK, I.Mogilnykh, V.Potapov, 2016)

Let (Γ, S) be an (r,s) pair. Let $T = (T_0, T_1)$ be a pair of disjoint nonempty independent vertex sets of Γ. The following are equivalent:

(a) *T is an S-bitrade.*

(b) *The function*

$$f^T(\bar{x}) = \chi_{T_0}(\bar{x}) - \chi_{T_1}(\bar{x}) = \begin{cases} (-1)^i & \text{if } \bar{x} \in T_i,\ i \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

is an eigenfunction corresponding to the eigenvalue $\theta = -r/s$.

(c) *The subgraph Γ^T of Γ induced by $T_0 \cup T_1$ is regular of degree $-\theta = r/s$ (since T_0 and T_1 are independent, Γ^T is bipartite).*

The weight distribution of an eigenfunction in a distance-regular graph

Lemma

The weight distribution

$$W(x) = \left(f(y), \sum_{y \in \Gamma_1(x)} f(y), \ldots, \sum_{y \in \Gamma_{\mathrm{diam}(\Gamma)}(x)} f(y) \right)$$

of an eigenfunction f of a distance-regular graph Γ equals $(f(x)W^i_{A,\theta})_{i=0}^{\mathrm{diam}(\Gamma)}$, where the coefficients $W^i_{A,\theta}$ depend on the intersection array $A = (b_0, \ldots, c_{\mathrm{diam}(\Gamma)})$ of Γ and the eigenvalue corresponding to f.

Corollary (w.d. (weight distribution) bound)

An eigenfunction f of a distance-regular graph has at least $\sum_{i=0}^{\mathrm{diam}(\Gamma)} |W^i_{A,\theta}|$ nonzeros.

Minimum trades

Theorem

Let Γ be a distance-regular graph, and let (Γ, S) be an (r, s) pair. Let $T = (T_0, T_1)$ be a pair of disjoint nonempty independent vertex sets of Γ. The following are equivalent:

(a') *T is a minimum (in the sence of the w.d. bound) S-bitrade.*

(b') *The function f^T is an eigenfunction of Γ corresponding to $-r/s$ whose number of nonzeros coinsides with the w.d. bound*

(c') *The subgraph Γ^T is a regular isometric subgraph of degree r/s.*

Moreover, in this case, (Γ, S) is a Delsarte pair, $-r/s$ is the smallest eigenvalue of Γ, and Γ^T is a distance-regular graph.

Latin bitrade

- The Hamming graph $H(n,q)$. The vertex set is the set $\{0,\dots,q-1\}^n$ of n-words over the alphabet $\{0,\dots,q-1\}$. Two words are adjucent if they differ in exactly one position. $H(n,2)$ is known as the n-cube.
- The clique designs in the Hamming graphs $H(n,q)$ are known as the latin hypercubes of order q (in the coding theory, distance-2 MDS codes); the clique clique bitrades are known as the latin bitrades [2]. The most known case $n = 3$ corresponds to the latin squares, see, e.g., [3].
- The bipartite distance-regular subgraph that corresponds to the minimum bitrade is the n-cube $H(n,2)$.

[2] V. N. Potapov. Multidimensional Latin bitrades, *Sib. Math. J.* 54(2) 317–324, 2013.

[3] N. J. Cavenagh. The theory and application of latin bitrades: A survey. *Math. Slovaca* 58(6) 691–718, 2008.

The number $L(n, q)$ of latin hypercubes

- If the order $q \geq 4$ is fixed, then

$$2^{2^{c' n + o(n)}} \leq L(n, q) \leq 2^{2^{c'' n + o(n)}}$$

- [V.Potapov, DK, P.Sokolova, 2008, 2011].
$\psi(x_1, \psi(x_2, \psi(x_3, \ldots \psi(x_{n-1}, x_n)\ldots)))$

ψ :

0	1	4	5	6	7	2	3
1	0	5	4	7	6	3	2
6	7	2	3	0	1	4	5
7	6	3	2	1	0	5	4
2	3	6	7	4	5	0	1
3	2	7	6	5	4	1	0
4	5	0	1	2	3	6	7
5	4	1	0	3	2	7	6

1	8	4	5	6	7	2	3	0
8	0	5	4	7	6	3	2	1
6	7	3	8	0	1	4	5	2
7	6	8	2	1	0	5	4	3
2	3	6	7	5	8	0	1	4
3	2	7	6	8	4	1	0	5
4	5	0	1	2	3	7	8	6
5	4	1	0	3	2	8	6	7
0	1	2	3	4	5	6	7	8

The number $L(n, q)$ of latin hypercubes

- If the dimension $n \geq 2$ is fixed, then

$$L(n, q) = 2^{O(q^n \log q)}.$$

- Lower bound, $n \geq 3$: [V.Potapov, arXiv:1510.06212]
- Important: not only the number of independent bitrades plays role, but also the number of ways to choose them.
- Upper bound: $\left((1 + o(1))\frac{q}{e^n}\right)^{\frac{q^n}{6}}$ [4]

[4] N.Linial, Z.Luria. An Upper Bound on the Number of High-Dimensional Permutations, *Combinatorica* 34(4) 471–486, 2014

Steiner systems

- The Johnson graph $J(v,k)$. The vertex set is the set of k-subsets of $\{1,2,\dots,v\}$. Two k-subsets are adjucent iff they intersect in $k-1$ elements. Assume $2k \le n$ ($J(n,k)$ and $J(n,n-k)$ are isomorphic).
- The clique designs in $J(v,k)$ are known as the Steiner systems $\mathrm{S}(k-1,k,v)$, the clique bitrades are known as the Steiner (bi)trades $\mathrm{T}(k-1,k,v)$. The bipartite distance-regular subgraph that corresponds to the minimum bitrade is the k-cube $H(k,2)$. The cardinality of the minimum bitrade: 2^k [5].

$$
\begin{array}{ccccccc}
 & & \{1,2,6\} & - & \{1,5,6\} & & \\
 & / & & \times & & \backslash & \\
\{1,2,3\} & - & \{1,5,3\} & & \{4,2,6\} & - & \{4,5,6\} \\
 & \backslash & & \times & & / & \\
 & & \{4,2,3\} & - & \{4,5,3\} & &
\end{array}
$$

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5 H.L.Hwang. On the structure of (v,k,t) trades. *J. Stat. Plann. Inference* 13:179–191, 1986.

The number of Steiner systems

- The number of Steiner triple systems $S(2,3,v)$: $2^{O(v^2 \log v)}$ More tightly,

$$\left(\frac{v}{e^2 3^{3/2}}\right)^{\frac{v^2}{6}} \leq |S(2,3,v)| \leq \left((1+o(1))\frac{v}{e^2}\right)^{\frac{v^2}{6}}$$

  lower bound: [6] upper bound: [7]
- The number of Steiner quadruple systems $S(3,4,v)$: $2^{O(v^3 \log v)}$ [V.N.Potapov, arXiv:1606.02426].
- $S(t,k,v)$ exist for large v [P.Keevash, arXiv:1401.3665]

[6] R.M.Wilson. Nonisomorphic Steiner triple systems, *Math Z* 135(4) 303–313, 1974.

[7] N.Linial, Z.Luria. An Upper Bound on the Number of Steiner Triple Systems, *Random Struct. Alg.* 43(4) 399–406, 2013

q-ary (subspace) Steiner systems

- Let V be a v-dimensional space over the finite field $GF(q)$. The Grassmann graph $J_q(n,d)$ is defined on the set of all k-dimensional subspaces of V. Two k-dimensional subspaces are adjucent iff they intersect in a $(k-1)$-dimensional subspace.
- All k–dimensional subspaces that include a fixed $(k-1)$–dimensional subspace, form a maximum clique ($v \geq 2k$).
- A clique design is a *q-ary Steiner system* $S_q[k-1,k,v]$, a subspace analog of $S(k-1,k,v)$. At this moment, only $\mathrm{S}_2[2,3,13]$ are known [M.Braun, at al. ArXiv: 1304.1462] for $k>2$.
- [DK,I.Mogilnykh,V.Potapov,2016] The subgraph corresponding to a minimum bitrade has the parameters of the dual polar graph $D_d(q)$ of order $\prod_{i=0}^{d-1}(q^i+1)$.

Halved hypercube

- The vertices of the halved n-cube are the binary n-words with even number of ones (i.e., a part of the bipartite n-cube $H(n,2)$). Two words are adjucent iff thew differ in exactly two positions.
- A maximum clique is the set of n words that differ in exactly one position from a fixed wodr with odd number of ones.
- A clique design in the halved n-cube is an extended 1-perfect code. Such codes exist iff n is a power of two.
- Bitrades exist for every even n. The minimum cardinality of a bitrade is $2^{n/2}$, e.g. $\{(x,x) \mid x \in \{0,1\}^{n/2}\}$. The corresponding graph is $H(n/2,2)$.

Transversals of a latin square

- A latin square graph is constructed on the q^2 cells of a latin $q \times q$ square. Two cells are adjucent iff they are in the same row, same column, or contain the same values.

0	1	2	3	4	5	6	7	8
1	**2**	0	**4**	5	3	7	8	6
2	0	**1**	5	3	4	8	6	7
3	4	5	**6**	7	8	0	1	**2**
4	5	3	7	**8**	6	1	2	0
5	3	4	8	6	**7**	2	0	1
6	7	8	0	1	2	**3**	4	5
7	8	6	1	2	0	4	**5**	3
8	**6**	7	2	0	1	5	3	**4**

- The cliques correspond to the rows, columns, and values.
- A clique design is a transversal of a latin square.
- The minimum cardinality of a bitrade is 6. The corresponding subgraph is $K_{3,3}$.

The number of transversals of a latin square

The logarithm of the maximum number of transversals in a latin square of order q is $\frac{q}{6}(\ln q + O(1))$. [8]

0	1	2	3	4	5	6	7	8
1	**2**	0	**4**	5	3	7	8	6
2	0	**1**	5	3	4	8	6	7
3	4	5	**6**	7	8	0	1	**2**
4	5	3	7	**8**	6	1	2	0
5	3	4	8	6	**7**	2	0	1
6	7	8	0	1	2	**3**	4	5
7	8	6	1	2	0	4	**5**	3
8	**6**	7	2	0	1	5	3	**4**

[8] V.Potapov. On the number of transversals in latin squares *Discr. Appl. Math.* 202:194–196, 2016

Other types of trades

- 1-perfect codes, $q > 2$.
- other designs, e.g., $S(t, k, v)$, $t < k - 1$.
- MDS codes with distance > 2 (systems of orthogonal latin hypercubes)
- orthogonal arrays

In all cases, the objects are connected with eigenfunctions, but not only corresponding to the smallest eigenvalue.

The minimum number of nonzeros of an eigenfunction

- The problem of finding minimum bitrades motivate us to study the minimum number of nonzeros of an eigenfunction of a given graph.
- $H(n, q > 2)$ — ??? (known: for the largest and for the smallest eigenvalues)
- $2(q-1)q^{n-2}$ for the second largest eigenvalue [A.Valyuzhenich, arXiv:1512.02606]