The lit-only σ-game and relevant mathematics

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Outline

Lit-only σ-game

Let D be a digraph, namely a pair of finite sets $(\mathrm{V}_D, \mathrm{A}_D)$ with $\mathrm{A}_D \subseteq \mathrm{V}_D \times \mathrm{V}_D$.

A configuration is a subset of V_D.

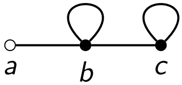
Let x be a configuration and v a vertex from V_D. If $v \in x$, then the configuration x can go to the configuration

$$x \triangle \mathrm{N}_D^+(v),$$

where $\triangle$ is the symmetric difference, and $\mathrm{N}_D^+(v)$ is the set of out-neighbours of v in D. The dynamical system in which the phase transition follows the above rule is called the lit-only σ-game.

Note that $(2^{\mathrm{V}_D}, \triangle)$ is a binary linear space.

An example



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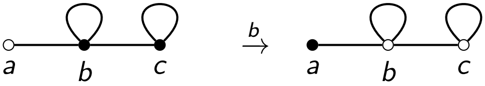

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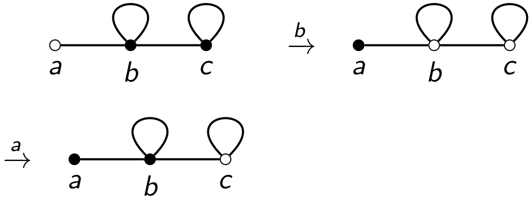

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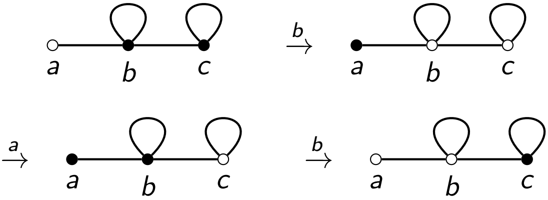

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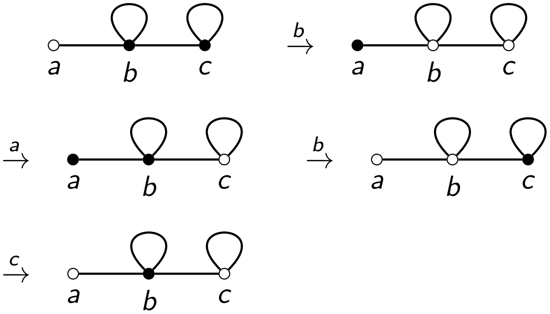

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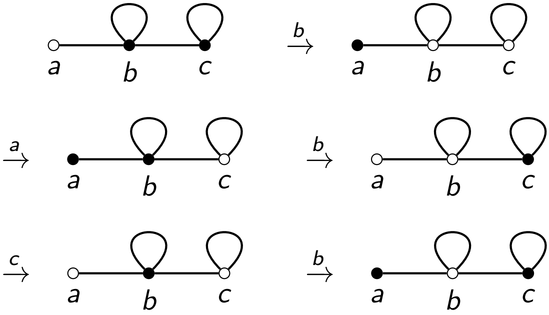

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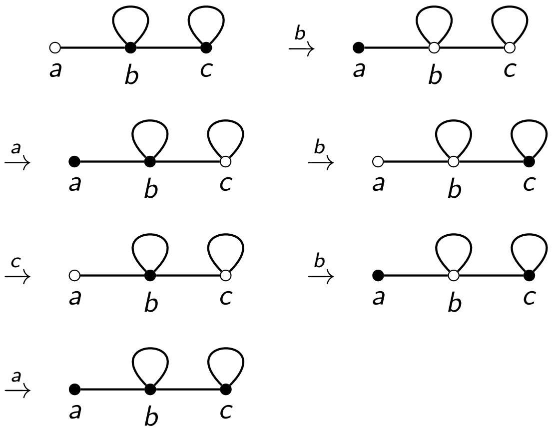

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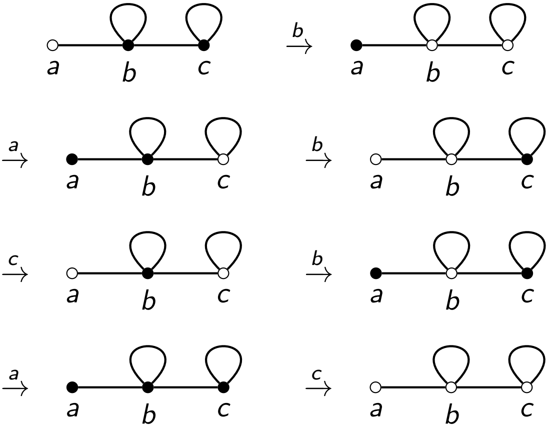

All-loops-on can reach all-off in the lit-only σ-game: $\mathsf{L}_D \xrightarrow{*} \emptyset$.

The lit-only σ-game reachability problem

THE PROBLEM: Take a digraph D and $x, y \in 2^{V_D}$. Is it true that $x \xrightarrow{*}_D y$?

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Theorem 1

If the digraph D is symmetric, then the above problem is polynomial time solvable.

Lit-only monoid

Let D be a digraph. For every $v \in \mathsf{V}_D$, construct a map $\mathcal{T}_v \in \mathrm{End}(2^{\mathsf{V}_D})$:

$$x \mapsto \begin{cases} x \triangle \mathsf{N}_D^+(v), & v \in x, \\ x, & v \notin x. \end{cases}$$

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Definition 2

The lit-only monoid of D is the multiplicative monoid generated by $\mathcal{T}_v$'s:

$$\textsc{Lom}_D := \langle \mathcal{T}_v : \ v \in \mathsf{V}_D \rangle.$$

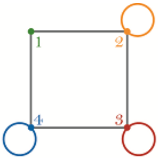
When D is loopless, we call $\textsc{Lom}_D$ the lit-only group of D, denoted by $\textsc{Log}_D$.

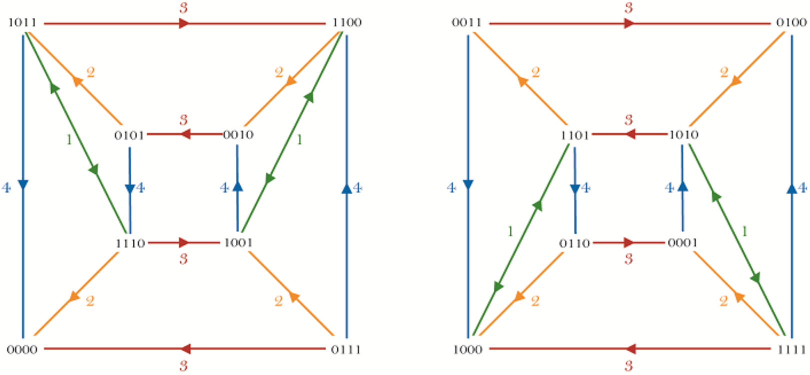
Phase space

The phase space of the lit-only σ-game on a digraph D is the digraph $\mathcal{PS}_D$ with:

- $\mathsf{V}_{\mathcal{PS}_D} = 2^{\mathsf{V}_D}$;
- $\mathsf{A}_{\mathcal{PS}_D} = \{(x, \mathcal{T}_v(x)) \colon\ v \in x \in 2^{\mathsf{V}_D}\}$.

Phase space: An example





The lit-only σ-game reachability problem, Contd.

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In general, as a dynamical system problem, we want to understand the action of LOM_D on 2^{V_D}.

Cayley digraph and its lit-only variant

Taking a group G, a set S and a map $\beta \in G^S$, the Cayley digraph $\Gamma(G, \beta)$ has vertex set G and arc set $\{g \to g\beta(s) : \ s \in S, g \in G\}$.

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Minicourse I: The lit-only restriction is due to faulty links!
Minicourse II: Does it make sense to consider the Cayley isomorphism problem with the lit-only restriction?

$\mathcal{PS}_D$ as a lit-only Cayley digraph

The power set of V, namely 2^V, forms an abelian group where the sum $x + y$ of two elements x and y of 2^V is given by the symmetric difference $x \triangle y$ of them. Let α be the map from V to 2^V such that $\alpha(v) = \{x \in 2^V : \ v \in x\}$.

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Digraphs D on vertex set V are just single variable functions β from V to $G = 2^V$.

- $D \Rightarrow \beta$: $\beta(v) \doteq \mathrm{N}_D^+(v) \in G, \forall v \in V$.
- $\beta \Rightarrow D$: $\mathrm{A}_D \doteq \{v \to w : \; w \in \beta(v)\}$.

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Assuming that digraph D and map $\beta \in G^V$ correspond to each other under the above bijection, then

$$\Gamma(2^V, \beta, \alpha) = \mathcal{PS}_D\,.$$

What is the difference made by the lit-only restriction?

Let

$$\Gamma(2^V, \beta) = \mathcal{PS}_D^\sigma,$$

which surely contains

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as a subgraph.

Playing the σ-*game* on the digraph D is to find a "good" path in $\mathcal{PS}_D^\sigma$ while playing the lit-only σ-game on D is to find a "good" path in $\mathcal{PS}_D$.

The reachability problem for $\mathcal{PS}_D^\sigma$ based on information of β (D) amounts to solving a system of linear equations over the binary field and hence is easy.

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The reachability problem for $\mathcal{PS}_D^{\sigma}$ based on information of β (D) amounts to solving a system of linear equations over the binary field and hence is easy. Our main work is to tell the difference between $\mathcal{PS}_D^{\sigma}$ and $\mathcal{PS}_D$.

Vector spaces associated to graphs

- Arc space: 2^{A_D}.
- Two subspaces of 2^{A_D} that appear often in algebraic graph theory: Cut space and cycle space.
- Vertex space: 2^{V_D}.
- Neighbour space: $\mathcal{N}_D := \langle \mathsf{N}_D^+(v) : v \in \mathsf{V}_D \rangle \leq 2^{\mathsf{V}_D}$.

The cosets of the neighbour space $\mathcal{N}_D$ in the vertex space is the set of all strongly connected components of $\mathcal{PS}_D^\sigma$. We may say that the difference between $\mathcal{PS}_D^\sigma$ and $\mathcal{PS}_D$ is small if we can show that these cosets are almost always still strongly connected components of $\mathcal{PS}_D$.

We call the dimension of $\mathcal{N}_D$ the rank of D, and we often use r for this number.

A question

Are there more general groups G, connection maps β, and lit-only restrictions α for which we can test efficiently the reachability on the lit-only Cayley digraph $\Gamma(G, \beta, \alpha)$ based on the local information of β and α, as possible generalization of Theorem 1?

The classification

According to the types of differences between $\mathcal{PS}_D^\sigma$ and $\mathcal{PS}_D$, we try to classify all strongly connected digraphs. We are successful for symmetric digraphs and have corresponding conjecture for asymmetric digraphs.

Prior to the classification results, we present our newly-discovered graph classes.

The natural maps from $\mathbb{N}$ to $\{0, 1\}$

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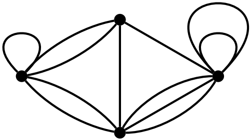
$$n \mapsto n \bmod 2.$$

The natural maps from multigraphs to graphs

A multigraph (symmetric nonnegative integer matrix):



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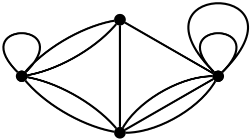

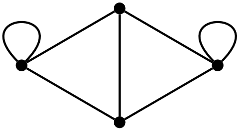

(a) Over the Boolean semifield $\mathbb{B}$

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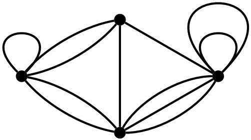

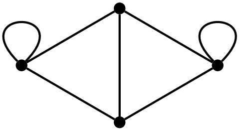

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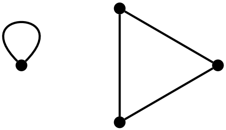

(b) Over the binary field $\mathbb{F}_2$

Line graphs

Given a multigraph G, its line graph, denoted by $\mathfrak{L}(G)$, is the graph with:

- Vertex set: $\mathsf{V}_{\mathfrak{L}(G)} = \mathsf{E}_G$;
- Edge set: $\mathsf{E}_{\mathfrak{L}(G)} = \{\{e, f\} : |\partial_G(e) \cap \partial_G(f)| \equiv 1, e, f \in \mathsf{E}_G\}$.

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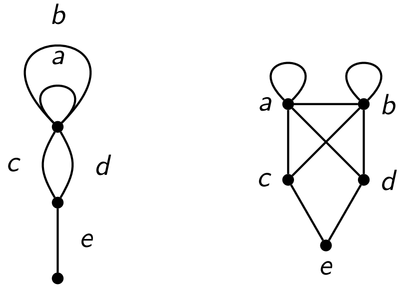

(a) G **(b)** $\mathfrak{L}(G)$

Figure: A multigraph and its line graph.

Characterization of loopless ordinary line graphs

Theorem 3 (Beineke's characterization)

A loopless graph is an ordinary line graph if and only if it does not contain any of the nine graphs below as a vertex-induced subgraph.

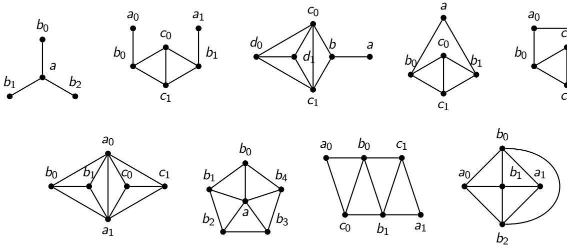

Figure: Nine forbidden vertex-induced subgraphs for loopless ordinary line graphs.

Three loopless extraordinary line graphs

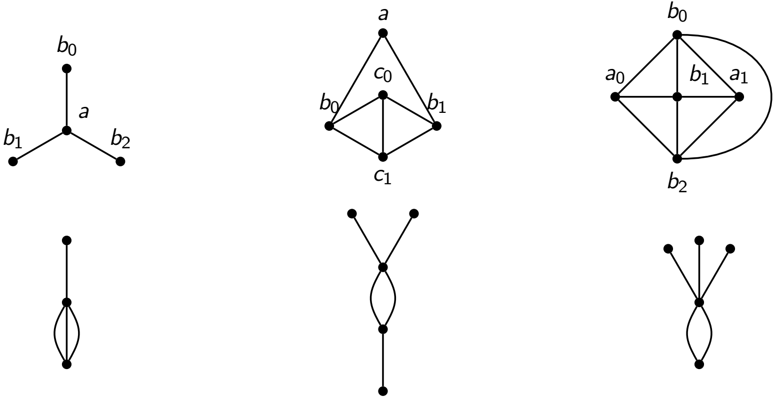

Figure: Three line graphs from the list of Beineke and their root multigraphs.

Quadratic form

Let $\mathbb{V}$ be a binary linear space. For any map $Q : \mathbb{V} \to \mathbb{F}_2$, we define its *polarisation* to be the map $\nabla_Q : \mathbb{V} \times \mathbb{V} \to \mathbb{F}_2$ such that

$$\nabla_Q(x, y) := Q(x + y) - Q(x) - Q(y), \quad \forall x, y \in \mathbb{V} .$$

A quadratic form on $\mathbb{V}$ is a map $Q : \mathbb{V} \to \mathbb{F}_2$ such that $Q(0) = 0$ and that ∇_Q is $\mathbb{F}_2$-bilinear, and so an alternating form ($\nabla_Q(x, x) = 0$).

Let X be a set.

Example 4

It is clear that $Q = \frac{|\cdot|}{2}$ is a quadratic form on $V = \binom{X}{\text{even}}$ and its polarisation ∇_Q is given by $\nabla_Q(x, y) \equiv |x \cap y|$ for all $x, y \in \binom{X}{\text{even}}$.

Example 5

We often write an element $\{A, B\} \in 2^X/\{\mathbf{0}, X\}$ as $A|B$ and thus view $2^X/\{\mathbf{0}, X\}$ as the set of splits on X. For each $S \in 2^X/\{\mathbf{0}, X\}$, we define $\overline{S_x}$ and S_x to be two sets such that $x \in S_x$, $x \notin \overline{S_x}$ and $S = S_x|\overline{S_x}$. For any $x \in X$, define the quadratic form Q_x on $V = 2^X/\{\mathbf{0}, X\}$ by setting $Q_x(S) \equiv |\overline{S_x}|$. Note that the polarisation of Q_x vanishes everywhere.

There is a natural nondegenerate pairing of $\binom{X}{\text{even}}$ and $2^X/\{\mathbf{0}, \mathsf{V}_G\}$ as $\mathbb{F}_2$-vectorspaces:

$$\binom{X}{\text{even}} \times 2^X/\{\mathbf{0}, \mathsf{V}_G\} \to \mathbb{F}_2 : (Z, A|B) \mapsto |A \cap Z|.$$

But it is not clear if there is also some natural correspondence between the quadratic forms on $2^{\mathsf{V}_G}/\{\mathbf{0}, \mathsf{V}_G\}$ and those on $\binom{\mathsf{V}_G}{\text{even}}$.

Euler form

Regarding a graph G as a one-dimensional abstract simplicial complex, the Euler characteristic of G (over $\mathbb{F}_2$), denoted by $\chi(G)$, is $|\mathsf{V}_G| - |\mathsf{E}_G| \in \mathbb{F}_2$.

The Euler form of a graph G (over $\mathbb{F}_2$), denoted by χ_G, is the quadratic form on 2^{V_G} given by

$$\chi_G(x) :\equiv \chi(G[x]) \equiv \sum_{v \in \mathsf{V}_G \setminus \mathsf{L}_G} v^*(x)^2 - \sum_{uv \in \mathsf{E}_G \setminus \mathsf{L}_G} u^*(x) v^*(x),$$

where $G[x]$ is the vertex-induced subgraph of G for subset $x \subseteq \mathsf{V}_G$.

Some graph classes

Opal line graphs: Line graphs of multigraphs with odd number of vertices.

Emerald line graphs: Line graphs of multigraphs with even number of vertices.

Cuspidal graphs: Graphs that are not line graphs.

Polished graphs: Graphs G that admit a (unique) quadratic form q_G on $\mathcal{N}_G$ such that $q_G \circ \mathrm{N}_G = \chi_G$. For polished graphs G, the Arf invariants of q_G determine the shape of $\mathcal{PS}_G$.

Unpolished graphs: Graphs G that are not polished.

Loop-linked digraphs: Digraphs D in which every vertex can go to a loop vertex and every vertex can be reached by a loop vertex.

Loopless digraphs: Digraphs D without loops.

Theorem 6

Let G be a loopless connected multigraph. Then, $\mathfrak{L}(G)$ is polished if and only if $|V_G| \not\equiv 2 \pmod{4}$.

Phase space of nonempty loopless strongly connected graphs (Theorem)

Graph classes	Phase space	
1. Opal line graphs	Neighbour space	$\bullet\ \mathbf{0}$ $\quad \binom{r+1}{2} \quad \binom{r+1}{4} \cdots \binom{r+1}{r-2} \quad \binom{r+1}{r}$
2. Polished emerald line graphs	Neighbour space	$\bullet\ \mathbf{0}$ $\quad \binom{r+2}{2} \quad \binom{r+2}{4} \cdots \binom{r+2}{r/2-1} \quad \binom{r+2}{r/2+1}$
of rank ≥ 4	Exception	$\binom{r+2}{1} \quad \binom{r+2}{3} \cdots \binom{r+2}{r/2-2} \quad \binom{r+2}{r/2}$
3. Unpolished emerald line graphs	Neighbour space	$\bullet\ \mathbf{0}$ $\quad \binom{r+2}{2} \quad \binom{r+2}{4} \cdots \binom{r+2}{r/2-2} \quad \binom{r+2}{r/2}$
of rank ≥ 4	Exception	$\binom{r+2}{1} \quad \binom{r+2}{3} \cdots \binom{r+2}{r/2-1} \quad \binom{r+2}{r/2+1}$
4. Polished cuspidal graphs, I	Neighbour space	$\bullet\ \mathbf{0}$ $\quad 2^{r-1}-2^{r/2-1} \quad 2^{r-1}+2^{r/2-1}-1$
5. Polished cuspidal graphs, II	Neighbour space	$\bullet\ \mathbf{0}$ $\quad 2^{r-1}-2^{r/2-1}-1 \quad 2^{r-1}+2^{r/2-1}$
6. Unpolished cuspidal graphs	Neighbour space	$\bullet\ \mathbf{0}$ $\quad 2^{r}-1$
	Exception	$2^{r-1}-2^{r/2-1} \quad 2^{r-1}+2^{r/2-1}$

Phase space of loop-linked strongly connected graphs (Theorem)

Graph classes	Phase space	
7. Loop-linked line graphs	Neighbour space	$\mathbf{0} \bullet \leftarrow \binom{r}{1} \leftarrow \binom{r}{2} \leftarrow \cdots \leftarrow \binom{r}{r-2} \leftarrow \binom{r}{r-1} \leftarrow \bullet \mathsf{L}_D$
8. Loop-linked cuspidal graphs	Neighbour space	$\mathbf{0} \bullet \leftarrow 2^r - 2 \leftarrow \bullet \mathsf{L}_D$

Phase space of loopless strongly connected asymmetric digraphs (Conjecture)

Graph class	Phase space	
9. Loopless asymmetric digraphs	Neighbour space	$\bullet$ $\mathbf{0}$ $(2^r - 1)$

Phase space of loop-linked strongly connected asymmetric digraphs (Conjecture)

Graph class	Phase space	
10. Asymmetric digraphs, I	Neighbour space	$\mathbf{0} \leftarrow \binom{r}{1} \leftarrow \binom{r}{2} \leftarrow \cdots \leftarrow \binom{r}{r-2} \leftarrow \binom{r}{r-1} \leftarrow \mathsf{L}_D$
11. Asymmetric digraphs, IIA	Neighbour space	$\mathbf{0} \leftarrow \binom{r+1}{2} \leftarrow \binom{r+1}{4} \leftarrow \cdots \leftarrow \binom{r+1}{r-2} \leftarrow \binom{r+1}{r}$
	Exception	$\binom{r+1}{1} \leftarrow \binom{r+1}{3} \leftarrow \cdots \leftarrow \binom{r+1}{r-3} \leftarrow \binom{r+1}{r-1} \leftarrow \mathsf{L}_D$
12. Asymmetric digraphs, IIB	Neighbour space	$\mathbf{0} \leftarrow \binom{r+1}{2} \leftarrow \binom{r+1}{4} \leftarrow \cdots \leftarrow \binom{r+1}{r-3} \leftarrow \binom{r+1}{r-1} \leftarrow \mathsf{L}_D$
	Exception	$\binom{r+1}{1} \leftarrow \binom{r+1}{3} \leftarrow \cdots \leftarrow \binom{r+1}{r-2} \leftarrow \binom{r+1}{r}$
13. Asymmetric digraphs, IIIA	Neighbour space	$\mathbf{0} \leftarrow 2^r - 2 \leftarrow \mathsf{L}_D$
14. Asymmetric digraphs, IIIB	Neighbour space	$\mathbf{0} \leftarrow 2^r - 1$
	Exception	$2^r - 1 \leftarrow \mathsf{L}_D$

Topaz groups

Definition 7

Let D be a loopless digraph. The Topaz group of D, denoted by TG_D, is the restriction of LOG_D to the invariant subspace $\mathcal{N}_D$.

Topaz groups

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Let D be a loopless digraph. The Topaz group of D, denoted by TG_D, is the restriction of $\textsc{Log}_D$ to the invariant subspace $\mathcal{N}_D$.

Theorem 8

Let D be a nonempty loopless strongly connected digraph. Then, TG_D is determined by $\mathcal{N}_D$ and the type of D as follow.

1. *Opal line graphs:* $\mathrm{Sym}_{\dim \mathcal{N}_D+1}$.
2. *Emerald line graphs of rank* ≥ 4*:* $\mathrm{Sym}_{\dim \mathcal{N}_D+2}$.
3. *Polished cuspidal graphs:* $\mathrm{O}^+(\mathcal{N}_D)$ *or* $\mathrm{O}^-(\mathcal{N}_D)$.
4. *Unpolished cuspidal graphs:* $\mathrm{Sp}(\mathcal{N}_D)$.
5. *Asymmetric digraphs:* $\mathrm{SL}(\mathcal{N}_D)$.

Lit-only groups

Theorem 9

Let D be a nonempty loopless strongly connected graph. Then, $\textsc{Log}_D$ is determined by $\mathcal{N}_D$ and the type of D as follow.

1. *Opal line graphs:* $\mathrm{Sym}_{\dim \mathcal{N}_D+1} \ltimes \mathcal{N}_D^{\operatorname{codim} \mathcal{N}_D}$.
2. *Emerald line graphs of rank* ≥ 4*:* $\mathrm{Sym}_{\dim \mathcal{N}_D+2} \ltimes \mathcal{N}_D^{\operatorname{codim} \mathcal{N}_D-1}$.
3. *Polished cuspidal graphs:* $\mathrm{O}^+(\mathcal{N}_D) \ltimes \mathcal{N}_D^{\operatorname{codim} \mathcal{N}_D}$ *or* $\mathrm{O}^-(\mathcal{N}_D) \ltimes \mathcal{N}_D^{\operatorname{codim} \mathcal{N}_D}$.
4. *Unpolished cuspidal graphs:* $\mathrm{Sp}(\mathcal{N}_D) \ltimes \mathcal{N}_D^{\operatorname{codim} \mathcal{N}_D-1}$.

Conjecture 10

Let D be a strongly connected loopless asymmetric digraph. Then, $\textsc{Log}_D$ is $\mathrm{SL}(\mathcal{N}_D) \ltimes \mathcal{N}_D^{\operatorname{codim} \mathcal{N}_D}$.

Characterization of loopless line graphs

Theorem 11

For a loopless graph G, the following statements are equivalent.

- *The graph G is a line graph.*
- *The graph G does not contain any graph in a set of thirty-two forbidden graphs as an induced subgraph.*

Characterization of loopless line graphs

Theorem 11

For a loopless graph G, the following statements are equivalent.

- *The graph G is a line graph.*
- *The graph G does not contain any graph in a set of thirty-two forbidden graphs as an induced subgraph.*
- *Every connected 6-vertex vertex-induced subgraph of G is a line graph.*
- *Every connected nonsingular 6-vertex vertex-induced subgraph of G is one of the eleven line graphs of 7-vertex trees.*

The thirty-two forbidden graphs

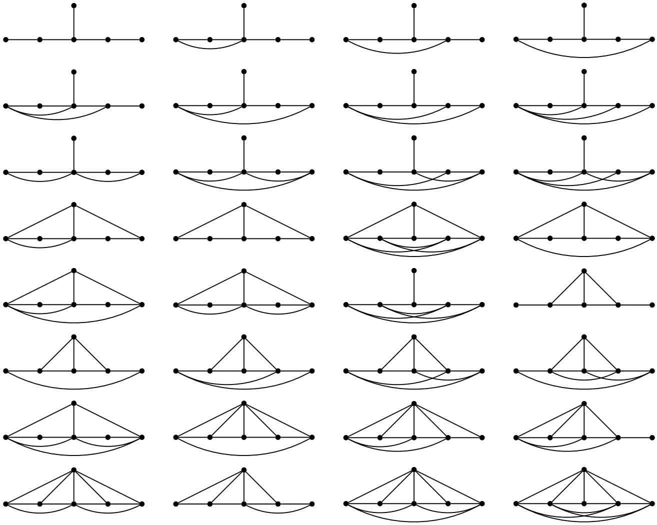


Figure: The 32 forbidden subgraphs for loopless line graphs.

The eleven line graphs of 7-vertex trees

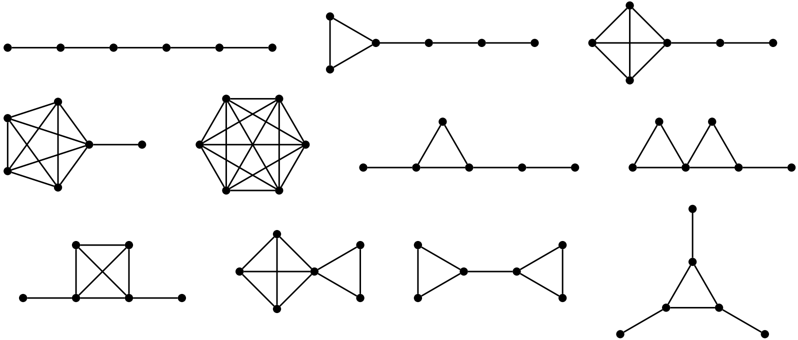


Figure: Eleven 6-vertex line graphs of trees.

Three observations

- There are 43 connected nonsingular 6-vertex graphs. They consist of the 32 forbidden graphs and the 11 line graphs of 7-vertex trees.
- All the 32 forbidden graphs are connected loopless polished cuspidal graphs with Arf invariant 1 and have a lit-only group isomorphic to $W(E_6)$.
- All those 11 line graphs are connected loopless opal line graphs and have Sym_7 as their lit-only group.

Edge clique partition

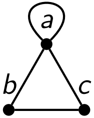
For a hypergraph $\mathcal{K} \subseteq 2^V$ with vertex set V, let $\mathrm{d}_{\mathcal{K}}(v)$ be $\{k \in \mathcal{K} \colon v \in k\}$ for all $v \in V$. The dual hypergraph of $\mathcal{K}$ is $\mathcal{K}^* = \{\mathrm{d}_{\mathcal{K}}(v) \colon v \in V\}$ on the vertex set $\mathcal{K}$.

Definition 12

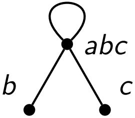
Let G be a graph without isolated vertices. An edge clique partition of G is a simple hypergraph $\mathcal{K} \subseteq 2^{\mathrm{V}_G} \setminus \{\emptyset\}$ such that the following conditions hold:

1. For every $v \in \mathrm{V}_G \setminus \mathrm{L}_G$, it holds $\mathrm{d}_{\mathcal{K}}(v) \in \binom{\mathcal{K}}{2}$;
2. For every $v \in \mathrm{L}_G$, it holds $\mathrm{d}_{\mathcal{K}}(v) \in \binom{\mathcal{K}}{1}$;
3. The edge set E_G is the disjoint union of $(k \cap \mathrm{L}_G) \cup \binom{k}{2}$ where k runs through all elements of $\mathcal{K}$;
4. For each $k \in \mathcal{K}$, it happens $|k \cap \mathrm{L}_G| \leq 1$.

Examples

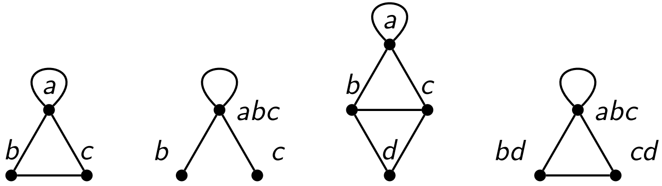


(a) G



(b) $\mathcal{K}_G^*$

Examples



(a) G **(b)** $\mathcal{K}_G^*$ **(c)** H **(d)** $\mathcal{K}_H^*$

Figure: Graphs, edge clique partitions and root graphs.

$$G = \mathfrak{L}(\mathcal{K}_G^*), \;\; H = \mathfrak{L}(\mathcal{K}_H^*)$$

Generalization of Whitney's theorem

Theorem 13 (Whitney's theorem)

A connected loopless ordinary line graph has an edge clique partition.

Generalization of Whitney's theorem

Theorem 13 (Whitney's theorem)

A connected loopless ordinary line graph has an edge clique partition.

Theorem 14 (Generalization of Whitney's theorem)

Except the four graphs, a connected ordinary line graph has a unique edge clique partition.

We can design a very simple linear-time algorithm to reconstruct the root graph of a line graph.

Graphs having two edge clique partitions

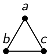
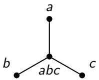
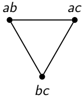
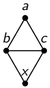
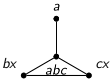
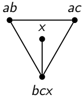
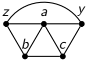
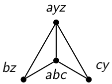
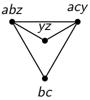
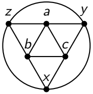
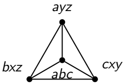
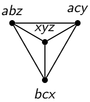
	Graph	Type A ECP	Type B ECP
$X = \emptyset$ $Y = \emptyset$ $Z = \emptyset$			
$X = \{x\}$ $Y = \emptyset$ $Z = \emptyset$			
$X = \emptyset$ $Y = \{y\}$ $Z = \{z\}$			
$X = \{x\}$ $Y = \{y\}$ $Z = \{z\}$			

Table: Graphs and the dual hypergraphs of their two edge clique partitions.

Characterization of loop-linked line graphs

Theorem 15

A loop-linked graph is a line graph if and only if it does not contain any graph below as a vertex-induced subgraph.

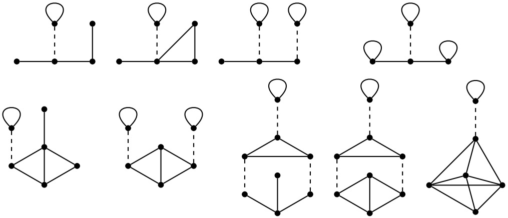


Figure: The 9 classes of forbidden vertex-induced subgraphs of loop-linked graphs. Dashed lines stand for paths of length zero or more.

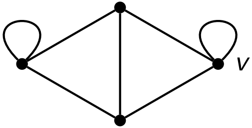
Box minus

Let G be a graph and v be a loop vertex of G.

Box minus

Let G be a graph and v be a loop vertex of G.

Let $G \boxminus v$ be the graph with vertex set $V_G \setminus \{v\}$ and edge set $E_G \triangle \binom{N_G(v)}{1} \triangle \binom{N_G(v)}{2}$.



(a) G

Box minus

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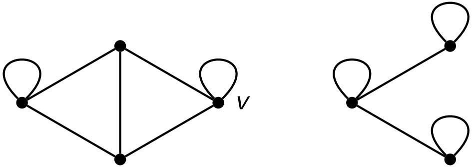
Let $G \boxminus v$ be the graph with vertex set $\mathsf{V}_G \setminus \{v\}$ and edge set $\mathsf{E}_G \triangle \binom{\mathsf{N}_G(v)}{1} \triangle \binom{\mathsf{N}_G(v)}{2}$.



(a) G **(b)** $G \boxminus v$

Another characterization of loop-linked line graphs

Theorem 16

A loop-linked graphs is cuspidal if and only if it can be reduced to one of the following graphs by a sequence of $-$ and $\boxminus$ operations through loop-linked graphs.

Figure: The two minimal (with respect to operators $-$ and $\boxminus$) loop-linked cuspidal graphs.

Summary

- Loopless line graphs have 32 forbidden subgraphs.

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- Most line graphs have unique edge clique partitions.

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- Loop-linked cuspidal graphs can be reduced to two graphs.
- Most line graphs have unique edge clique partitions.

Problem 17

Why do they happen?

A double covering from $\mathcal{PS}_G$ to $\mathcal{PS}_{G\boxminus v}$: $wv \in \mathsf{E}_G$

$$\phi_0(\alpha)|_{\mathsf{V}_{G\boxminus v}} = \alpha, \phi_0 + \phi_1 = \mathsf{N}_G(v), v^* \circ \phi_i = i, \forall \alpha \in 2^{\mathsf{V}_{D\boxminus v}}, i \in \mathbb{F}_2 .$$

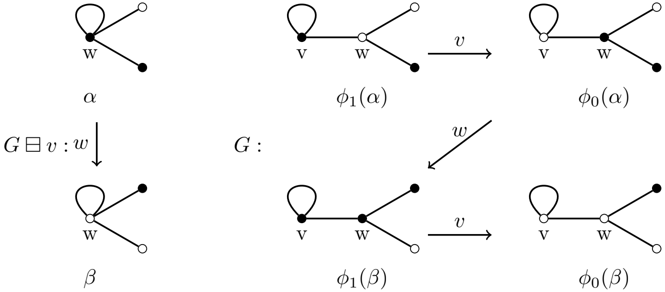

Figure: For $w \in \mathsf{N}_G(v) \setminus \{v\}$, a w-arc in $\mathcal{PS}_{G\boxminus v}$ is lifted to two length-two walks in $\mathcal{PS}_G$.

A double covering from $\mathcal{PS}_G$ to $\mathcal{PS}_{G \boxminus v}$: $wv \notin \mathsf{E}_G$

$$\phi_0(\alpha)|_{\mathsf{V}_{G \boxminus v}} = \alpha, \phi_0 + \phi_1 = \mathsf{N}_G(v), v^* \circ \phi_i = i, \forall \alpha \in 2^{\mathsf{V}_{D \boxminus v}}, i \in \mathbb{F}_2 \,.$$

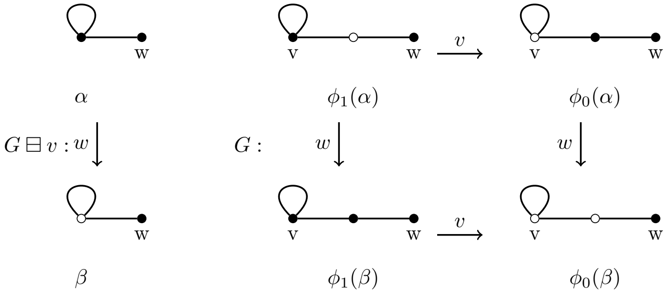

Figure: For $w \in \mathsf{V}_G \setminus \mathsf{N}_G(v)$, a w-arc in $\mathcal{PS}_{G \boxminus v}$ is lifted to two w-arcs in $\mathcal{PS}_G$.

A generalization of Sutner's Theorem

Theorem 18 (Sutner's Theorem)

$\mathsf{L}_G \in \mathcal{N}_G$.

A generalization of Sutner's Theorem

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Theorem 19 (A generalization of Sutner's Theorem)

$\mathsf{L}_G \xrightarrow{*}_G \mathbf{0}$.

A generalization of Sutner's Theorem

Theorem 18 (Sutner's Theorem)

$\mathsf{L}_G \in \mathcal{N}_G$.

Theorem 19 (A generalization of Sutner's Theorem)

$\mathsf{L}_G \xrightarrow{*}_G \mathbf{0}$.

Proof.

Induct on the number of vertices.
Pick a loop vertex $v \in \mathsf{L}_G$. By induction hypothesis, $\mathsf{L}_{G \boxminus v} \xrightarrow{*}_{G \boxminus v} \mathbf{0}$.

$$\mathsf{L}_G = \phi_1(\mathsf{L}_{G \boxminus v}) \xrightarrow{v}_G \phi_0(\mathsf{L}_{G \boxminus v}) \xrightarrow{*}_G \phi_0(\mathbf{0}) = \mathbf{0}\,. \qquad \square$$