Journal of Algebra **218,** 412–435 (1999)
Article ID jabr.1999.7882, available online at http://www.idealibrary.com on IDEAL®

Y-groups via Transitive Extension

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Communicated by Walter Feit

Received September 29, 1998

Motivated by an earlier observation by J. H. Fischer, around 1980 B. Conway conjectured that the Coxeter diagram Y_{555} on Fig. 1 together with a single additional (so-called "spider") relation $(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1$ form a presentation for the wreath product of the Monster group M and a group of order 2. This conjecture was proved by S. P. Norton and the author in 1990. The original proof was rather involved, relying on simple connectedness results for certain diagram geometries, on numerous data obtained by coset enumeration on a computer, and on some delicate calculations with subgroups coming from the 26-node theorem. In the present work we follow an inductive approach to the identification of Y-groups by considering larger Y-groups as transitive extensions of smaller ones. Along these lines we obtain an alternative identification proof for the Y-groups which is almost computer-free: we refer to only one result of (double) coset enumeration. Our approach provides a uniform understanding of the Y-groups, particularly of features such as centres and redundant generators.

1. INTRODUCTION

We start with the Coxeter diagram given in Fig. 1 known as the Y_{555}-diagram and the following relation known as the spider relation:

$$(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1.$$

* This paper has been written while the author was visiting RIMS at University of Kyoto. Warm hospitality of this institution is gladly acknowledged.

0021-8693/99 $30.00

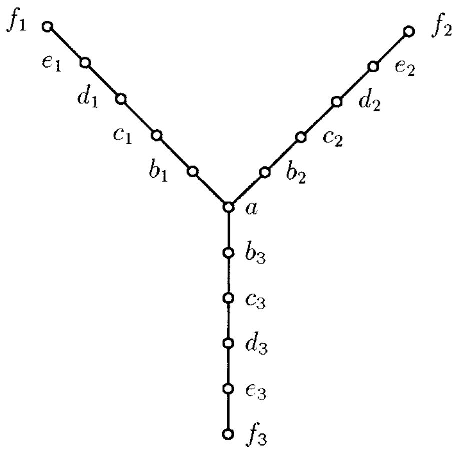

FIGURE 1

For $2 \leq p, q, r \leq 5$ define Y_{pqr} to be the quotient over the spider relation of the Coxeter group, whose generators are a and

first p terms from b_1, c_1, d_1, e_1, f_1;
first q terms from b_2, c_2, d_2, e_2, f_2;
first r terms from b_3, c_3, d_3, e_3, f_3;

and whose Coxeter relations correspond to the subdiagram of the Y_{555}-diagram induced by the generators. A homomorphic image of the group Y_{pqr} is called a Y_{pqr}-group and the Coxeter generators of Y_{pqr} are usually identified with their images in a Y_{pqr}-group. If Z is a Y_{pqr}-group and $x, y, \ldots, z$ are some Coxeter generators of Z (or rather of Y_{pqr}) then $Z\lfloor x, y, \ldots, z\rfloor$ denotes the subgroup in Z generated by all its Coxeter generators except for $x, y, \ldots, z$. In these terms if x is the terminal node of the left arm of the Coxeter diagram of Y_{pqr} and $p \geq 3$ then $Y_{pqr}\lfloor x\rfloor$ is a $Y_{(p-1)qr}$-group.

If $\min\{p, q, r\} < 2$ then we define Y_{pqr} as $Y_{p_1q_1r_1}\lfloor x, \ldots, z\rfloor$ where $p_1 = \min\{2, p\}$, $q_1 = \min\{2, q\}$, $r_1 = \min\{2, r\}$, and $x, \ldots, z$ are the nodes in the Coxeter diagram of $Y_{p_1q_1r_1}$ whose removal gives the Coxeter diagram of Y_{pqr}. Suppose that $p - 1, q, r \geq 2$ and that x is the terminal node of the left arm of the Coxeter diagram of Y_{pqr}. Then a Y_{pqr}-group Z is said to be *strong* if $Z\lfloor x\rfloor \cong Y_{(p-1)qr}$.

If $p, q, r \geq 2$ then every defining relation of Y_{pqr} has even length which implies the following.

LEMMA 1.1. *Suppose that $2 \leq p, q, r \leq 5$, that Z is a Y_{pqr}-group and that $O^2(Z) = Z$. Then the direct product of Z and a group of order 2 is also a Y_{pqr}-group.*

The structure of the groups Y_{pqr} is given in Table I. The groups above Y_{442} have been identified by means of coset enumeration on a computer in [CNS88], the group Y_{442} has been identified by D. Z. Djoković also by coset enumeration on a computer and a computer-free identification can be found in [CP92]. The group Y_{333} has been identified using double coset enumeration performed by S. A. Linton (cf. [Lin89; Soi91]). The isomorphism type of Y_{443} was proved by combining the results in [Nor90; Nor92; Iv91; Iv92a] (see also [Con92]). The group Y_{433} has been identified in [Iv92b]. It has been proved in [Soi89] that the isomorphism $Y_{443} \cong 2 \times M$ implies the isomorphism $Y_{44} \cong M \wr 2$. An independent characterization of Fischer groups as Y-groups can be found in [Vi97]. The groups Y_{p22}, $p \geq 5$, were identified in [Pr89] with certain orthogonal groups over $GF(3)$ (we do not present these results here). If $q \geq 3$, $r \geq 2$ then $Y_{5qr} = Y_{4qr}$; Y_{632} and higher Y-groups collapse to a group of order 2 (cf. Subsection 3.5 of the present paper). It is worth mentioning that Y-groups map isomorphically onto their natural images in Y_{444} except for the groups Y_{421} and Y_{422} which are losing their centres of order 2.

TABLE I

pqr	Y_{pqr}	$[Y_{pqr} : Y_{(p-1)qr}]$
321	$2 \times \mathrm{Sp}_6(2)$	56
421	$2 \cdot \Omega_8^+(2):2$	240
331	$2^7.(2 \times \mathrm{Sp}_6(2))$	128
431	$2 \times \mathrm{Sp}_8(2)$	255
441	$\Omega_{10}^-(2):2$	528
222	$3^5 : \Omega_5(3):2$	243
322	$2 \times \Omega_7(3)$	728
422	$2 \cdot \Omega_8^+(3):2$	2160
332	$2 \times 2 \cdot \mathrm{Fi}_{22}$	28,160
432	$2 \times \mathrm{Fi}_{23}$	31,671
442	$3 \cdot \mathrm{Fi}_{24}$	920,808
333	$2 \times 2^2 \cdot {}^2E_6(2)$	2,370,830,336
433	$2 \times 2 \cdot BM$	27,143,910,000
443	$2 \times M$	97,239,461,142,009,186,000
444	$M \wr 2$	$\lvert M \rvert \sim 10^{54}$

2. FROM Y-GROUPS TO Y-GRAPHS

We start this section with a definition. Let Δ be a graph and G be a vertex- and edge-transitive automorphism group of Δ. Let Ξ be another graph and H be an automorphism group of Ξ which is also assumed to be vertex- and edge-transitive. For a vertex $\alpha \in \Xi$ let $\Xi_i(\alpha)$ denote the set of vertices at distance i from α and let $H(\alpha)$ denote the stabilizer of α in H. Then (Ξ, H) is said to be *weakly locally* (Δ, G) if for every $\alpha \in \Xi$ there is an isomorphism

$$\varphi_\alpha : (\Delta, G) \to (\Xi_1(\alpha), H(\alpha)),$$

of permutation groups such that whenever $\{x, y\}$ is an edge of Δ, $\{\varphi_\alpha(x), \varphi_\alpha(y)\}$ is an edge of Ξ. Notice that if (Ξ, H) is weakly locally (Δ, G) then H is a transitive extension of G (cf. [Su86, p. 545]). Identifying Δ and $\Xi_1(\alpha)$ via φ_α we can say that the subgraph in Ξ induced by $\Xi_1(\alpha)$ is a union of the orbitals of the action of G on Δ and this union contains the orbital formed by the edges of Δ. When H and G are clear from the context we simply say that Ξ is weakly locally Δ.

Suppose that Z is a Y_{pqr}-group, where $p \geq 2$, that x is the terminal node of the left arm of the Coxeter diagram of Y_{pqr} and y is the node adjacent to x. We have fixed the left arm to simplify the notation. Define a Y-graph $\Gamma = \Gamma(Z, x)$ to be a graph on the set of right cosets in Z of the subgroup $Z\lfloor x \rfloor$ in which two cosets $Z\lfloor x \rfloor g_1, Z\lfloor x \rfloor g_2$ are adjacent if there is an element h_1 in the former coset and an element h_2 in the latter coset such that $h_2 = xh_1$. In other terms the edges of Γ are the images under the natural action of Z of the pair $e := \{Z\lfloor x \rfloor, Z\lfloor x \rfloor x\}$. If $Z(e)$ is the elementwise stabilizer of the edge e then

$$Z(e) = Z\lfloor x \rfloor \cap Z\lfloor x \rfloor^x.$$

It is obvious that the latter group contains $Z\lfloor x, y \rfloor$ and the Y-graph Γ is called *correct* if $Z(e) = Z\lfloor x, y \rfloor$.

Let $\alpha = Z\lfloor x \rfloor$, $\beta = Z\lfloor x \rfloor x$, $\gamma = Z\lfloor x \rfloor xy$, $H = \langle x, y \rangle \cong \mathrm{Sym}_3$ and suppose that Γ is correct. Then $Z\lfloor x \rfloor = Z(\alpha)$ acts on $\Gamma_1(\alpha)$ as it acts on the cosets of $Z\lfloor x, y \rfloor$. Furthermore, since $(xy)^3 = 1$ and $y \in Z\lfloor x \rfloor$ we have

$$\gamma \cdot x = Z\lfloor x \rfloor xyx = Z\lfloor x \rfloor yxy = Z\lfloor x \rfloor xy = \gamma,$$

which shows that $T := \{\alpha, \beta, \gamma\}$ is a triangle in Γ on which H induces the natural action. The images of T under Z are called *Y-triangles*. Thus the action of $Z(\alpha)$ on $\Gamma_1(\alpha)$ is similar to its action on the vertex set of $\Delta := \Gamma(Z\lfloor x \rfloor, y)$ and two vertices in $\Gamma_1(\alpha)$ are adjacent whenever the corresponding vertices in Δ are adjacent. This shows that $\Gamma(Z, x)$ is weakly locally $\Gamma(Z\lfloor x\rfloor, y)$ (notice that $Z\lfloor x\rfloor$ is a $Y_{(p-1)qr}$-group). We summarize the most important case of this observation in

LEMMA 2.1. *Suppose that Z is a strong Y_{pqr}-group where $p-1, q, r \geq 2$ and that $\Gamma(Z, x)$ is correct. Then $\Gamma(Z, x)$ is weakly locally $\Gamma(Y_{(p-1)qr}, y)$.*

Suppose that both $\Gamma(Z, x)$ and $\Gamma(Y_{pqr}, x)$ are correct. This is the case, for instance, when $\Gamma(Z, x)$ is correct and $Y_{(p-1)qr}\lfloor y\rfloor$ is a maximal subgroup in $Y_{(p-1)qr}$. Then the natural homomorphism

$$\varphi: Y_{pqr} \to Z$$

induces a covering

$$\psi: \Gamma\big(Y_{pqr}, x\big) \to \Gamma(Z, x),$$

of graphs such that the Y-triangles are contractible with respect to ψ, which gives

LEMMA 2.2. *Suppose that Z is a strong Y_{pqr}-group and that both $\Gamma(Z, x)$ and $\Gamma(Y_{pqr}, x)$ are correct. Suppose further that the Y-triangles in $\Gamma(Z, x)$ generate the fundamental group of $\Gamma(Z, x)$. Then $Z \cong Y_{pqr}$.*

In some cases examples of Y-groups can be constructed via their Y-graphs.

LEMMA 2.3. *Let y be the terminal node of the left arm of the $Y_{(p-1)qr}$-diagram, where $p-1, q, r \geq 2$, and z be the node adjacent to y. Let Ξ be a graph and Z be a vertex- and edge-transitive automorphism group of Ξ and suppose that the following conditions hold for α being a vertex of Ξ:*

(i) *$\Gamma(Y_{(p-1)qr}, y)$ is correct;*

(ii) *(Ξ, Z) is weakly locally $(\Gamma(Y_{(p-1)qr}, y), Y_{(p-1)qr})$ and φ_α is the corresponding isomorphism;*

(iii) *if $\beta = \varphi_\alpha(Y_{(p-1)qr}\lfloor y\rfloor)$ then the setwise stabilizer in Z of $\{\alpha, \beta\}$ is the direct product of $Z(\alpha) \cap Z(\beta)$ and a group of order 2 generated by x;*

(iv) *the setwise stabilizer in $Y_{(p-1)qr}$ of $\{Y_{(p-1)qr}\lfloor y\rfloor, Y_{(p-1)qr}\lfloor y\rfloor y\}$ is the direct product $\langle y\rangle \times Y_{(p-1)qr}\lfloor y, z\rfloor$ and $\langle y\rangle$ is the centre of this stabilizer.*

Then Z is a strong Y_{pqr}-group.

Proof. The Coxeter generators of Z are x and the set K of (the images under φ_α of) the Coxeter generators of $Y_{(p-1)qr}$. By (ii) the generators in K satisfy the Coxeter relations and the spider relation. By (iii) x commutes with all the generators in K except for y. The product xy induces an action of order 3 on the triangle $T = \{\alpha, \beta, \gamma\}$ where $\gamma = \varphi_\alpha(Y_{(p-1)qr}\lfloor y\rfloor y)$.

Hence $\sigma := (xy)^3$ is in the elementwise stabilizer L of this triangle. By (i) $L \cong Y_{(p-1)qr}\lfloor y, z \rfloor$, by (iii), (iv) we have $\sigma \in Z(L)$ and $Z(L)$ is trivial by (iv). Hence $(xy)^3 = 1$ and the result follows. ∎

3. AN INDUCTIVE APPROACH

Our general approach to Y_{pqr} is the following. We consider a group Z acting vertex and edge transitively on a graph Ξ and we show eventually that Ξ is $\Gamma(Y_{pqr}, x)$ where x is the terminal node of the left arm of the Y_{pqr}-diagram. First we show that Ξ is weakly locally $\Gamma(Y_{(p-1)qr}, y)$ where y is the node adjacent to x. Then we check the conditions in Lemma 2.3 and conclude that Z is a strong Y_{pqr}-group. Finally we show that the Y-triangles generate the fundamental group of Γ and conclude from Lemma 2.2 that $Z \cong Y_{pqr}$. On the last step we use the following sufficient condition of triangulability which is a straightforward generalization of Lemma 5 from [Ron81].

LEMMA 3.1. *Let Ξ be a graph of diameter d and suppose that for every $2 \leq i \leq d$ the following two conditions hold*:

(i) *if $\beta \in \Xi_i(\alpha)$ then the subgraph in Ξ induced by $\Xi_1(\alpha) \cap \Xi_{i-1}(\beta)$ is connected*;

(ii) *if $\beta, \gamma \in \Xi_i(\alpha)$ and $\beta \in \Xi_1(\gamma)$ then the distance in Ξ between $\Xi_1(\alpha) \cap \Xi_{i-1}(\beta)$ and $\Xi_1(\alpha) \cap \Xi_{i-1}(\gamma)$ is at most* 1.

Then Ξ is triangulable which means that its fundamental group is generated by the triangles.

In some cases we are able to show that the covering of Ξ under consideration induces another covering of graphs whose bijectivity is known from the literature. For this we use the strategy introduced in [Iv94].

3.1. $Y_{222} \cong 3^5 : \Omega_5(3) : 2$

The Coxeter group C of the diagram Y_{222} is affine of type E_6, so that C is the semidirect product of the E_6-lattice L and the spherical Coxeter group of type E_6, isomorphic to $\Omega_5(3) : 2$. Let σ be the expression in the brackets of the spider relation. Then the image of σ in C/L is of order 10 and hence $\sigma^{10} \in L$. Direct calculations in the E_6-lattice show that the normal closure of σ^{10} generates $3L$ and the image in $L/3L \cong 3^6$ of this closure is one dimensional, hence the result.

There is an orthogonal form on $O_3(Y_{222})$ and $Y_{222}/O_3(Y_{222})$ is the full automorphism group of this form. Then $\Gamma(Y_{222}, c_1)$ is a graph on the set of all vectors in a five-dimensional $GF(3)$-space W with a non-singular orthogonal form, such that $v, w \in W$ are adjacent if $(v+w)$ is a plus vector, which means that the orthogonal complement $(v+w)^{\perp}$ contains a two-dimensional totally singular subspace. It is straightforward to calculate that the suborbit diagram of $\Gamma(Y_{222}, c_1)$ is in Fig. 2.

3.2. $Y_{322} \cong 2 \times \Omega_7(3)$

Let W be a seven-dimensional $GF(3)$-space with a non-singular quadratic form and $Z \cong 2 \times \Omega_7(3)$ be the full automorphism group of this form. Let Ξ be a graph on the set of non-zero isotropic vectors in W in which two such vectors are adjacent if their inner product is plus 1. Direct calculations show that the suborbit diagram is in Fig. 3.

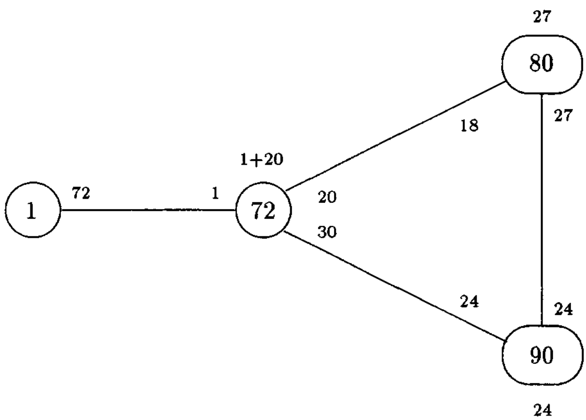

FIGURE 2

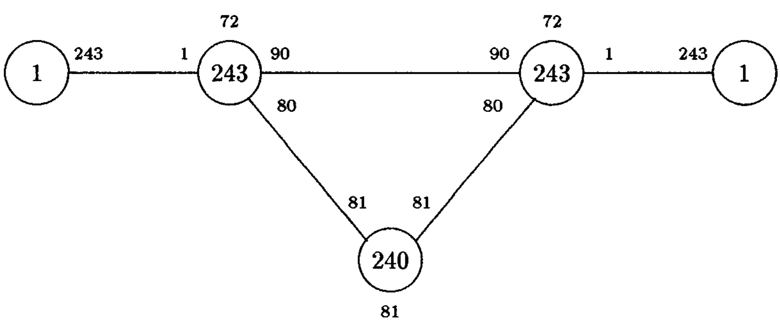

FIGURE 3

If $\alpha \in \Xi$ then $Z(\alpha) \cong 3^5 : \Omega_5(3):2 \cong Y_{222}$ and $O_3(Z(\alpha))$ acts regularly on $\Xi_1(\alpha)$ which immediately shows that Ξ is weakly locally $\Gamma(Y_{222}, c_1)$. It is easy to check that the remaining conditions in Lemma 2.3 are also satisfied and hence Z is a strong Y_{322}-group. From Fig. 3 we see that all triangles in Ξ are Y-triangles. Direct calculations in the orthogonal module W enable one to check the conditions in Lemma 3.1. Thus Ξ is triangulable, hence $Z \cong Y_{322}$ and $\Xi \cong \Gamma(Y_{322}, d_1)$ by Lemma 2.2.

For $\{i, j, k\} = \{1, 2, 3\}$ the nodes $a, b_i, c_i, d_i, b_j, c_j, b_k$ on the Y_{555}-diagram induce a spherical E_7-diagram, so that the corresponding Coxeter group is isomorphic to $\mathrm{Sp}_6(2) \times 2$ and its centre is generated by the following element (cf. [CNS88]):

$$f_{ijk} := \left(ab_ic_id_ib_jc_jb_k\right)^9.$$

For $i = 2$ and 3 put $X_i = Y_{322}\lfloor c_i \rfloor$ and let Σ_i be the subgraph in $\Gamma(Y_{322}, d_1)$ induced by the images of $Y_{322}\lfloor d_1 \rfloor$ under X_i. The Coxeter diagram of X_i is spherical of type E_7 and since all the Coxeter generators in Y_{322} are pairwise different, either $X_i \cong \mathrm{Sp}_6(2) \times 2$ or $X_i \cong \mathrm{Sp}_6(2)$. In the latter case $|\Sigma_i| = 28$ and X_i acts on Σ_i doubly transitively. By observing that $\Gamma(Y_{322}, d_1)$ does not contain cliques of size 28, or otherwise one concludes that $X_i \cong \mathrm{Sp}_6(2) \times 2$ and the suborbit diagram of Σ_i with respect to the action of X_i is in Fig. 4.

Comparing Fig. 4 with the diagram of $\Gamma(Y_{322}, d_1)$, we immediately deduce that $Z(X_2) = Z(X_3) = Z(Y_{322})$ and in terms of the above paragraph that $f_{123} = f_{132}$.

LEMMA 3.2. *If $q, r \geq 2$ then the element $f_{123} = f_{132}$ is in the centre of Y_{3qr}.*

Proof. A Coxeter generator of Y_{322} commutes with f_{1ij} since the latter element generates the centre of Y_{322}. On the other hand d_2 and higher terms clearly commute with f_{132} and the result follows. ∎

Permuting the indices p, q, r we obtain obvious analogues of Lemma 3.2 (compare the centres of Y-groups in Table I).

In Fig. 5 we present the suborbit diagram of $\Gamma(Y_{322}, c_2)$.

Notice that $\Gamma(Y_{322}, c_2)$ is the unique orbital graph of valency 288 of $\Omega_7(3)$ acting on the cosets of $\mathrm{Sp}_6(2)$ and every subgroup in $\Omega_7(3)$ of index 3159 is isomorphic to $\mathrm{Sp}_6(2)$.

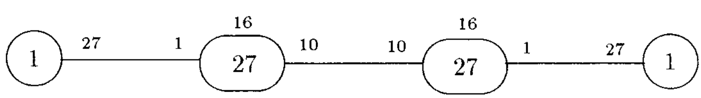

FIGURE 4

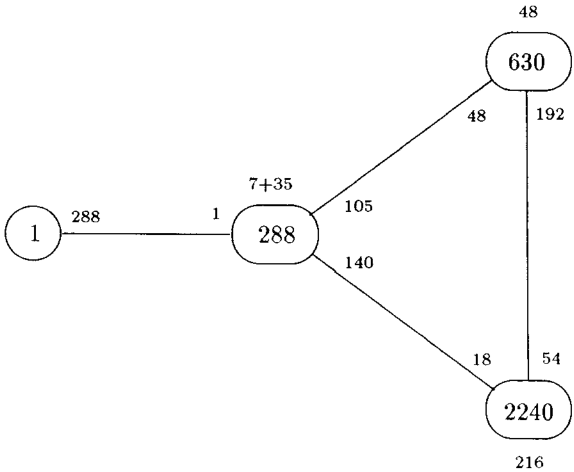

FIGURE 5

3.3. $Y_{422} \cong 2 \cdot \Omega_8^+(3) : 2$

Let W be an eight-dimensional $GF(3)$-space with a non-singular quadratic form of plus type. The automorphism group of this form is $2 \cdot \Omega_8^+(3) : 2^2$ (cf. [ATLAS]). Let Z be a subgroup of index 2 in the automorphism group which contains a subgroup $H \cong 2 \times \Omega_7(3)$ trivially intersecting the centre. Then $Z \sim 2 \cdot \Omega_8^+(3).2_2$ in the atlas notation. Let O be the orbit of Z on the set of non-isotropic vectors in W such that H stabilizes a vector from O and let Ξ be a graph on O in which two vectors are adjacent if their inner product is plus 1. Then the suborbit diagram of Ξ is in Fig. 6.

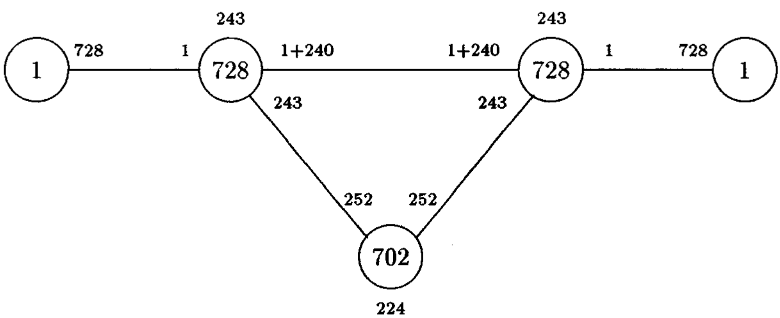

FIGURE 6

Performing some easy calculations in the orthogonal module W, we check that a triangle of Ξ is contained in 72 complete subgraphs on four vertices. In view of the suborbit diagram of $\Gamma(Y_{322}, c_1)$ this shows that Ξ is weakly locally $\Gamma(Y_{322}, c_1)$. It is straightforward to check the conditions in Lemma 2.3 and to conclude that Z is a strong Y_{422}-group. Finally the conditions in Lemma 3.1 hold, which show the isomorphism between Z and Y_{422}.

3.4. $Y_{332} \cong 2 \times 2 \cdot \mathrm{Fi}_{22}$

In what follows we deal with two graphs, we denote by $\Delta(\mathrm{Fi}_{23})$ and $\Delta(3 \cdot \mathrm{Fi}_{24})$. The former is the graph on the 3-transpositions in the Fischer group Fi_{23} in which two transpositions are adjacent if they commute; the latter is triple antipodal covering of the graph on 3-transpositions of the Fischer group Fi_{24} in which two transpositions are adjacent if they commute. The suborbit diagram of $\Delta(\mathrm{Fi}_{23})$ with respect to the action of Fi_{23} is in Fig. 7 while the suborbit diagram of $\Delta(3 \cdot \mathrm{Fi}_{24})$ with respect to the action of $3 \cdot \mathrm{Fi}_{24}$ is in Fig. 8. It was proved in [Ron81] that both $\Delta(\mathrm{Fi}_{23})$ and $\Delta(3 \cdot \mathrm{Fi}_{24})$ are triangulable (see also [Mei91] for a stronger characterization of these and related graphs).

Let us turn to Y_{332}. By Lemma 3.2 $\langle f_{213} \rangle$ is central in both Y_{332} and $Y_{332}\lfloor d_1 \rfloor$ and hence it is in the kernel of the action of Y_{332} on $\Gamma(Y_{332}, d_1)$. Consider the action of $Z := 2 \cdot \mathrm{Fi}_{22}$ (the non-split extension) on the cosets of a subgroup isomorphic to $\Omega_7(3)$. One of the orbital graphs (we denote it by Ξ) with respect to this action has the suborbit diagram in Fig. 9.

In view of the diagram and the remark at the end of Subsection 3.2, we conclude that Ξ is weakly locally $\Gamma(Y_{322}, c_2)$. Now it is easy to check the conditions in Lemma 2.3 and to conclude that Z is a strong Y_{332}-group.

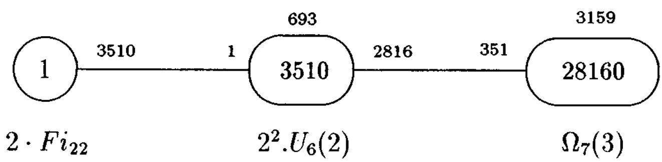

FIGURE 7

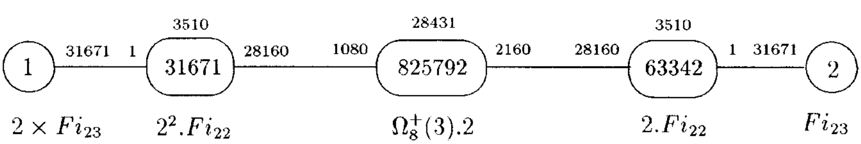

FIGURE 8

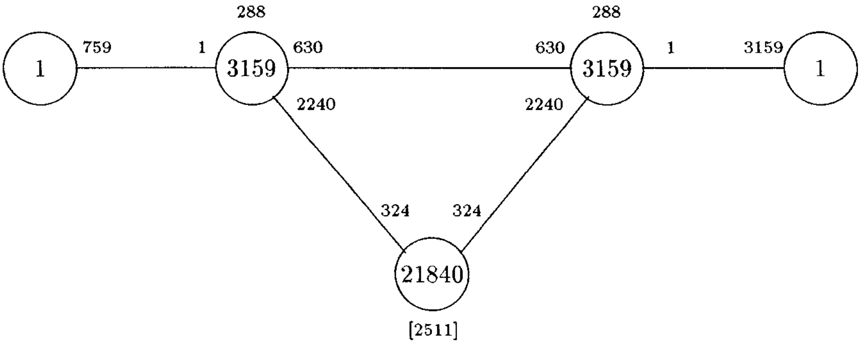

FIGURE 9

One can see from the suborbit diagram of $\Delta(\mathrm{Fi}_{23})$ that the stabilizer in Fi_{23} of a vertex $\alpha \in \Delta = \Delta(\mathrm{Fi}_{23})$ is isomorphic to Z and its actions on the vertex-set of Ξ and on $\Delta_2(\alpha)$ are similar. Furthermore the subgraph in Δ induced by $\Delta_2(\alpha)$ is also an orbital of valency 3159. Noticing that the stabilizer in Z of a triangle in Ξ is isomorphic to Sym_7 while the stabilizer in Fi_{23} of a triangle in $\Delta(\mathrm{Fi}_{23})$ is of the form $[2^{11}].U_4(2)$ (in particular it does not involve Sym_7), we have

LEMMA 3.3. *The subgraph in* Δ *induced by* $\Delta_2(\alpha)$ *and the graph* Ξ *with the suborbit diagram in Fig.* 9 *are two different orbitals of valency* 3159 *of the action of* $2 \cdot \mathrm{Fi}_{22}$ *on the cosets of* $\Omega_7(3)$.

Using Lemma 3.3 and calculating in the graph $\Delta(\mathrm{Fi}_{23})$ it is not difficult to check that the conditions in Lemma 3.1 are satisfied for Ξ, which gives the isomorphism $Y_{332}/\langle f_{213}\rangle \cong 2 \cdot \mathrm{Fi}_{22}$. Finally Lemma 1.1 completes the identification.

Noticing that the Coxeter diagram of Y_{331} is affine of type E_7, it is not difficult to identify $Y_{332}\lfloor c_3\rfloor/\langle f_{123}, f_{213}\rangle$ with a maximal subgroup in Fi_{22} of the form $2^6:\mathrm{Sp}_6(2)$. The subdegrees of Fi_{22} acting on the cosets of $2^6:\mathrm{Sp}_6(2)$, as calculated in [ILLSS], are

$$1, \quad 135, \quad 1260, \quad 2304, \quad 8640, \quad 10{,}080, \quad 45{,}360, \quad 143{,}360, \quad 241{,}920^2.$$

Since $Y_{332}\lfloor b_3, c_3\rfloor$ has index 2304 in $2^6:\mathrm{Sp}_6(2)$ the above subdegrees show that $\Gamma(Y_{332}, c_3)$ is correct and that it is isomorphic to the unique orbital graph of valency 2304 of the action of Fi_{22} on the cosets of $2^6:\mathrm{Sp}_6(2)$.

3.5. $Y_{432} \cong 2 \times \mathrm{Fi}_{23}$

Let $Z = \mathrm{Fi}_{23}$ and Ξ be the complement of $\Delta(\mathrm{Fi}_{23})$. Then the vertex stabilizer $Z(\alpha)$ of the action of Z on Ξ is isomorphic to $2 \cdot \mathrm{Fi}_{22}$ which is the index 2 commutator subgroup of Y_{332}. The suborbit diagram of Ξ is in

Fig. 10 and by Lemma 3.3 Ξ is weakly locally $\Gamma(Y_{332}, d_1)$. Checking the conditions in Lemma 2.3 we conclude that Z is a Y_{432}-group.

The natural homomorphism φ: $\overline{Y}_{432} := Y_{432}/\langle f_{213}\rangle \to Z$ induces a covering

$$\psi: \Gamma(Y_{432}, e_1) \to \Xi,$$

of graphs with respect to which the Y-triangles are contractible. Our nearest goal is to show that ψ induces a covering of $\Delta = \Delta(\mathrm{Fi}_{23})$.

Let $\tilde{P} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$ be a 2-path in $\Gamma(Y_{432}, e_1)$, $P = (s_1, s_2, s_3)$ be its image in Ξ and suppose that s_1 and s_3 are adjacent in Δ. Since ψ is a covering of graphs the stabilizer of $\tilde{P}$ in $\overline{Y}_{432}$ maps isomorphically onto the stabilizer H_1 of P in Z. On the other hand the suborbit diagram of Ξ and the remark at the end of Subsection 3.2 show that $H_1 \cong \mathrm{Sp}_6(2)$. Without loss of generality we assume that $\{\tilde{s}_1, \tilde{s}_2\} = \{Y_{432}\lfloor e_1\rfloor, Y_{432}\lfloor e_1\rfloor e_1\}$, so that $\{s_1, s_2\} = \{Z\lfloor e_1\rfloor, Z\lfloor e_1\rfloor e_1\}$. Let $\tilde{\Sigma}$ be the set of images of $\tilde{s}_1$ under $Y_{432}\lfloor d_2\rfloor$ and Σ be the set of images of s_1 under $Z\lfloor d_2\rfloor$. Comparing the isomorphism $Y_{422} \cong 2\cdot\Omega_8^+(3):2$ and the list of maximal subgroups in Fi_{23} or otherwise one concludes that $Z\lfloor d_2\rfloor \cong \Omega_8^+(3):2$ and hence $|\Sigma| = 1080$. Thus the restriction of ψ to $\tilde{\Sigma}$ is either a bijection or has fibers of size 2. In any case without loss of generality we can assume that $P \subseteq \Sigma$, $\tilde{P} \subset \tilde{\Sigma}$ and by the above sentence the stabilizer of $\{\tilde{s}_1, \tilde{s}_3\}$ in $\overline{Y}_{432}\lfloor d_2\rfloor$ has index at most 2 in the stabilizer H_2 of $\{s_1, s_3\}$ in $Z\lfloor d_2\rfloor$. From the suborbit diagram of $\Gamma(Y_{422}, e_1)$ we see that $H_2 \cong 2 \times 2\cdot U_4(3):2$. Thus the stabilizers of $\{\tilde{s}_1, \tilde{s}_3\}$ in $\overline{Y}_{432}$ contains a subgroup isomorphic to $\mathrm{Sp}_6(2)$ and a subgroup isomorphic to $2\cdot U_4(3)$. On the other hand the stabilizer in Fi_{23} of an edge in Δ, isomorphic to $2^2\cdot U_6(2)$ (non-split extension) is generated by any two of its subgroups isomorphic to $\mathrm{Sp}_6(2)$ and $2\cdot U_4(3)$. Hence the stabilizer of $\{\tilde{s}_1, \tilde{s}_3\}$ in $\overline{Y}_{432}$ maps isomorphically onto the stabilizer of $\{s_1, s_3\}$ in Z which shows that ψ induces a covering

$$\chi: \tilde{\Delta} \to \Delta,$$

of graphs. Here the vertex set of $\tilde{\Delta}$ is that of $\Gamma(Y_{432}, e_1)$ and the edges are the images of $\{\tilde{s}_1, \tilde{s}_3\}$ under Y_{432}. Since χ is a covering of graphs, the

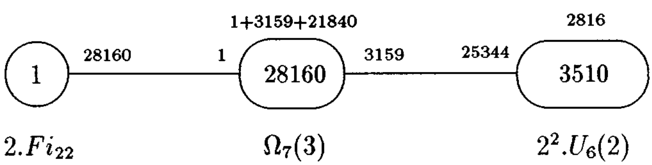

FIGURE 10

restriction of ψ to $\tilde{\Sigma}$ must be a bijection and hence χ induces an isomorphism of the subgraph in $\tilde{\Delta}$ induced by $\tilde{\Sigma}$ onto the subgraph in Δ induced by Σ. The latter graph is the antipodal folding of $\Gamma(Y_{422}, e_1)$ and it is of rank 3. Since this subgraph obviously contains triangles and Fi_{23} acts transitively on the set of triangles in Δ, we conclude that all the triangles in Δ are contractible with respect to χ. But then χ (and hence ψ as well) must be an isomorphism since the triangles generate the fundamental group of Δ [Ron81]. Application of Lemma 1.1 completes the identification of Y_{432}.

Analysing the list of maximal subgroups in Fi_{23} it is not difficult to identify $Y_{432}\lfloor c_3 \rfloor$ with $2 \times \mathrm{Sp}_8(2)$. Consider the action of Y_{432} on $\Gamma(Y_{432}, c_3)$. By Lemma 3.2 f_{231} is in the kernel of the action. The stabilizer in $\bar{Y}_{432}$ of the edge $e := \{Y_{432}\lfloor c_3 \rfloor, Y_{432}\lfloor c_3 \rfloor c_3\}$ obviously contains $\bar{Y}_{432}\lfloor b_3, c_3 \rfloor \cong \mathrm{Sym}_9$. On the other hand the subdegrees of the action of Fi_{23} on the cosets of $\mathrm{Sp}_8(2)$ were calculated in [ILLSS]. The only non-trivial subdegree which divides the index 130,560 of Sym_9 in $\mathrm{Sp}_8(2)$ is 13,056 and the corresponding 2-point stabilizer is isomorphic to Sym_{10}. Thus $\Gamma(Y_{432}, c_3)$ is not correct but in fact there is a way to "correct" the situation by adjoining an additional generator. Let $H \cong \mathrm{Sym}_{10}$ be the stabilizer in $\bar{Y}_{432}$ of the edge e. Then the subdiagram in Y_{432} which is the Coxeter diagram of $\bar{Y}_{432}\lfloor b_3, c_3 \rfloor$ can be extended to that of H by adjoining a node adjacent to e_1 or to d_2. Since f_{213} is in the centre of Y_{432} the extra node (denote it by f_1) must be adjacent to e_1. Since H has no outer automorphisms, f_1 commutes with c_3. We claim that f_1 also commutes with b_3. This claim can be checked by noticing that every edge of $\Gamma(Y_{432}, c_3)$ is contained in $210 = [\mathrm{Sym}_{10} : \mathrm{Sym}_6 \times \mathrm{Sym}_4]$ triangles (cf. [ILLSS]) and that b_3 is involved in the expression for the element f_{213} and the latter commutes with f_1. Thus Y_{432} is a Y_{532}-group. Furthermore, f_1 commutes with $Y_{432}\lfloor e_1 \rfloor \cong 2 \times 2 \cdot \mathrm{F}_{22}$, the latter subgroup is self-centralized in Y_{432} and by Lemma 3.2 its centre is $\langle f_{123}, f_{213} \rangle$. Since e_1 has product of order 3 with both f_1 and f_{123} we conclude that the latter two elements are equal.

LEMMA 3.4. $Y_{532} \cong Y_{432}$.

Proof. Suppose that $Y_{532}\lfloor f_1 \rfloor$ is a proper subgroup in Y_{532} and consider the action of $\bar{Y}_{532} := Y_{532}/\langle f_{231} \rangle$ on $\Gamma(Y_{532}, f_1)$. Then the structure of Y_{432}, Y_{332}, and Y_{232} show that the elementwise stabilizers of a vertex, an edge, and a triangle in $\bar{Y}_{543}$ are isomorphic to Fi_{23}, $2 \cdot \mathrm{Fi}_{22}$, and $\Omega_7(3)$, respectively. Hence $\Gamma(Y_{532}, f_1)$ is weakly locally the complement Ξ of $\Delta(\mathrm{Fi}_{23})$ with the suborbit diagram given in this subsection. If the diameter of $\Gamma(Y_{532}, f_1)$ is 1 then the action of Y_{532} on the vertex set of the graph is doubly transitive and it is an easy exercise to show that this is not possible. On the other hand from the suborbit diagram of Ξ we see that the number of vertices at distance 2 from a given vertex is at most

$$31671 \cdot 3510/25345 < 4500.$$

Comparing this estimate with the indices of maximal subgroups in Fi_{23} we conclude that there is only one vertex at distance 2. Since the action of Fi_{23} on Ξ is primitive, this gives a contradiction.

By the above lemma and the paragraph before it, we obtain

COROLLARY 3.5. *If $q \geq 3$ and $r \geq 2$ then $Y_{5qr} = Y_{4qr}$ and $f_1 = f_{123} = f_{132}$.*

Consider Y_{632} with the obvious meaning. Then the generator corresponding to the terminal node of the left arm of the Coxeter diagram commutes with f_{123} and has product of order divisible by 3 with f_1. By Corollary 3.5 this gives

COROLLARY 3.6. *Y_{632} and higher Y-groups collapse to a group of order 2.*

3.6. $Y_{442} \cong 3 \cdot \mathrm{Fi}_{24}$

Consider the action of $Z := 3 \cdot \mathrm{Fi}_{24}$ on $\Delta := \Delta(3 \cdot \mathrm{Fi}_{24})$. If $\alpha \in \Delta$ then $Z(\alpha) \cong 2 \times \mathrm{Fi}_{23} \cong Y_{432}$ and α can be identified with the unique non-trivial element in the centre of $Z(\alpha)$ (this element is an involution which maps onto a 3-transposition in Fi_{24}). In these terms if $\beta \in \Delta_i(\alpha)$ then the product $\alpha\beta$ is of order 2, 3, 6 and 3 for $i = 1, 2, 3$, and 4, respectively.

Let $\beta \in \Delta_2(\alpha)$. Then $Z(\alpha) \cap Z(\beta) \cong \Omega_8^+(3):2 \cong Y_{432}\lfloor d_2 \rfloor$. Since the commutator subgroup Z' still acts distance transitively on Δ, we conclude that $Z(\alpha) \cap Z(\beta)$ is not contained in the direct factor Fi_{23} of $Z(\alpha)$. Since all subgroups in Fi_{23} isomorphic to $\Omega_8^+(3):2$ are conjugate, this specifies the action of $Z(\alpha)$ on $\Delta_2(\alpha)$ and in particular shows that this action is similar to the action of Y_{432} on the vertex set of $\Gamma(Y_{432}, d_2)$. Since $Y_{432}\lfloor c_2, d_2 \rfloor \cong \Omega_8^+(2):2$ is a maximal subgroup of index 28,431 in $Y_{432}\lfloor d_2 \rfloor \cong \Omega_8^+(3):2$, we conclude that $\Gamma(Y_{432}, d_2)$ is correct of valency 28,431. The suborbit diagram of Δ shows that the subgraph in Δ induced by $\Delta_2(\alpha)$ is also an orbital of that valency. We claim that they are different orbitals. Indeed, by Lemma 3.5 the stabilizer in $Y_{432} = Y_{532}$ of a triangle in $\Gamma(Y_{432}, d_2)$ contains $Y_{532}\lfloor b_2, c_2, d_2 \rfloor \cong \mathrm{Sym}_9$ while the stabilizer in Z of a triangle in Δ is of the form $2^3.U_6(2)$ and does not involve Sym_9. Hence the claim follows.

Notice that the set

$$\Theta(\beta) = \{\gamma^{\alpha} | \gamma \in \Delta_2(\alpha) \cap \Delta_1(\beta)\} = \Delta_2(\alpha) \cap \Delta_1(\beta^{\alpha})$$

is an orbit of length 28,431 of $Z(\alpha) \cap Z(\beta)$ on $\Delta_2(\alpha)$ containing vertices which are at distance 2 from β in Δ. It follows from [PS97] that the action of Fi_{23} on the cosets of $\Omega_8^+(3):2$ has subdegree 28,431 with multiplicity 1, which means that if

$$\varphi: (\Gamma(Y_{432}, d_2), Y_{432}) \to (\Delta_2(\alpha)), Z(\alpha))$$

is an isomorphism of permutation groups which sends $Y_{432}\lfloor d_2 \rfloor$ onto β, then $\Theta(\beta)$ is the image under φ of the set of vertices adjacent to $Y_{432}\lfloor d_2 \rfloor$ in $\Gamma(Y_{432}, d_2)$. Thus we have

LEMMA 3.7. *Let Ξ be a graph on the set of vertices of $\Delta = \Delta(3 \cdot \mathrm{Fi}_{24})$ in which two vertices are adjacent if they are at distance 2 in Δ. Then Ξ is weakly locally $\Gamma(Y_{432}, d_2)$.*

Now it is easy to see that the conditions in Lemma 2.3 are satisfied and hence $Z \cong 3 \cdot \mathrm{Fi}_{24}$ is a Y_{442}-group (by Lemma 3.5 it is also a Y_{552}-group).

Our next goal is to show that the natural homomorphism

$$\varphi: Y_{442} \to Z$$

induces a covering of Δ. Let $\tilde{P} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$ be a 2-path in $\Gamma(Y_{442}, \lfloor e_1 \rfloor)$, $P = (s_1, s_2, s_3)$ be its image in Ξ and suppose that s_1 and s_3 are adjacent in Δ. Since φ induces a covering of $\Gamma(Y_{442}, e_1)$ onto Ξ, the stabilizer of $\tilde{P}$ in Y_{442} is isomorphic to $\Omega_8^+(2):2$ which is the stabilizer of P in Z. Let $\tilde{\Sigma}$ be the set of images of $Y_{442}\lfloor e_1 \rfloor$ (considered as a vertex of $\Gamma(Y_{442}, e_1)$) under $Y_{442}\lfloor e_2 \rfloor$ and let Σ be the set of images of $Z\lfloor e_1 \rfloor$ under $Z\lfloor e_2 \rfloor$. Since $Y_{442}\lfloor e_2 \rfloor \cong Z\lfloor e_2 \rfloor \cong Y_{432} \cong 2 \times \mathrm{Fi}_{23}$, $\tilde{\Sigma}$ maps bijectively onto Σ. Furthermore $Y_{442}\lfloor e_2 \rfloor$ acts on $\tilde{\Sigma}$ with kernel of order 2 and the induced action is isomorphic to that of Fi_{23} on the vertex set of $\Delta(\mathrm{Fi}_{23})$. Without loss of generality we assume that $\tilde{P} \subset \tilde{\Sigma}$ in which case it follows from the suborbit diagram of $\Delta(\mathrm{Fi}_{23})$ that the stabilizer of $\{\tilde{s}_1, \tilde{s}_3\}$ in $Y_{442}\lfloor e_2 \rfloor$ is of the form $2^3.U_6(2)$. Since the stabilizer of $\{s_1, s_3\}$ in Z, isomorphic to $2 \times 2 \cdot \mathrm{Fi}_{22}$ is generated by its subgroups isomorphic to $\Omega_8^+(2):2$ and $2^3.U_6(2)$, we conclude that the stabilizer of $\{\tilde{s}_1, \tilde{s}_3\}$ in Y_{442} maps isomorphically onto the stabilizer of $\{s_1, s_3\}$ in Z which implies that φ induces a covering

$$\chi: \tilde{\Delta} \to \Delta,$$

of graphs. The subgraph in $\tilde{\Delta}$ induced by $\tilde{\Sigma}$ maps isomorphically onto the subgraph in Δ induces by Σ and both these subgraphs are isomorphic to $\Delta(\mathrm{Fi}_{23})$. Since the latter graph contains triangles and Z acts transitively on the set of triangles in Δ, we conclude that the triangles are contractible with respect to χ. Since Δ is triangulable by [Ron81] both χ and ψ are isomorphisms and hence $Y_{442} \cong 3 \cdot \mathrm{Fi}_{24}$. Now analysing the maximal subgroups in Fi_{24} or otherwise one can check that $Y_{442}\lfloor c_3 \rfloor \cong \Omega_{10}^-(2):2$.

3.7. $Y_{333} \cong 2 \times 2^2 \cdot {}^2E_6(2)$

By Lemma 3.2 $\langle f_{123}, f_{213}, f_{312}\rangle$ is central in Y_{333} and $\langle f_{213}, f_{312}\rangle$ is contained in $Y_{333}\lfloor d_1 \rfloor$. Consider the action of $Z = {}^2E_6(2)$ on the cosets of $\mathrm{Fi}_{22} = Y_{233}/\langle f_{213}, f_{312}\rangle$. The intersection numbers of the centralizer algebra of this action have been calculated in [ISa96]. These calculations show particularly that there is an orbital graph Ξ of valency 694,980 with edge stabilizer isomorphic to a maximal subgroup of Fi_{22} isomorphic to $2^6 : \mathrm{Sp}_6(2)$. Furthermore, every edge of Ξ is in exactly 13,644 triangles. Since

$$13644 = 1260 + 2304 + 10080$$

is the only decomposition of the number of triangles on an edge into the lengths of suborbits of Fi_{22} on the cosets of $2^6 : \mathrm{Sp}_6(2)$ (compare Subsection 3.4), we conclude that Ξ is weakly locally $\Gamma(Y_{332}, c_3)$. It is easy to check the conditions in Lemma 2.3, using the information in [ISa96]. Hence Z is a Y_{333}-group. A possible way to identify Z with $Y_{333}/\langle f_{123}, f_{213}, f_{312}\rangle$ would be to show that the fundamental group of Ξ is generated by the Y-triangles. But this seems to be far too difficult, since the structure of Ξ is rather complicated and there are many classes of cycles in this graph. By this reason we refer to the original identification of Y_{333} which follows from the double coset enumeration performed by S. A. Linton (cf. [Lin89; Soi91]).

3.8. $Y_{433} \cong 2 \times 2 \cdot B$

Let $Z \cong 2 \cdot B$ be the extension of the Baby Monster by its Schur multiplier of order 2. Let Ω be the conjugacy class of involutions in Z with centralizers of the form $2^2 \cdot {}^2E_6(2)$ (cf. [ATLAS]). Then Z acts on Ω by conjugation, the centre of Z is the kernel and the induced action is similar to that of B on the cosets of $2 \cdot {}^2E_6(2)$. One of the orbitals of this action has the suborbit diagram as given in Fig. 11 (compare [Iv94]) and we denote this orbital also by Ω. The vertices of Ω can be identified with the involutions in the corresponding conjugacy class of $2 \cdot B$. Then the vertex α' antipodal to α is the product of α and the involution in the centre of Z. Furthermore the product $\alpha\beta$ has order 2, 2, 3, 4, 6 and 4 if β is contained in $\Omega_1(\alpha)$, $\Omega_1(\alpha')$, $\Omega_2^3(\alpha)$, $\Omega_2^4(\alpha)$, $\Omega_2^3(\alpha')$, and $\Omega_3^4(\alpha)$, respectively. Notice that by joining in Ω the antipodal vertices we obtain the graph isomorphic to the subgraph in Monster graph from the next subsection induced by the vertices adjacent to a given vertex.

The group $Z(\alpha) \cong 2^2 \cdot {}^2E_6(2)$ (the commutator subgroup of Y_{333}) acts on $\Omega_2^3(\alpha)$ as it acts on the cosets of its subgroup $2 \cdot \mathrm{Fi}_{22}$ (the commutator

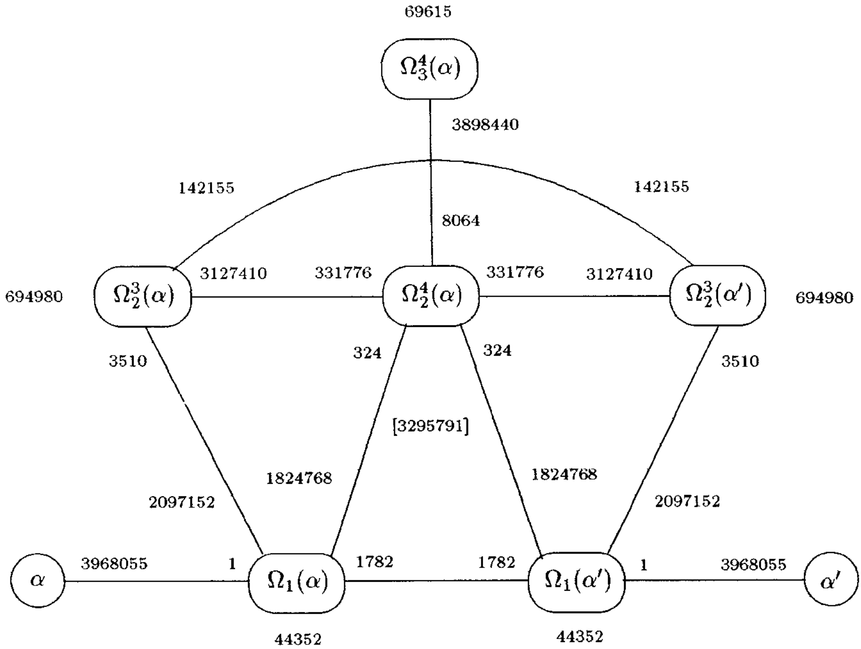

FIGURE 11

subgroup of Y_{332}). Hence this action is similar to the action induced by Y_{333} on $\Gamma(Y_{333}, d_1)$. The graph $\Gamma(Y_{333}, d_1)$ has valency 694,980 and one can see from the diagram of Ω that this is also the valency of the subgraph in Ω induced by $\Omega_2^3(\alpha)$. By [ISa96] the subdegree 694,980 appears with multiplicity 1 in the action of ${}^2E_6(2)$ on the cosets of Fi_{22}. This means that for $\beta \in \Omega_2^3(\alpha)$ the subgroup $Z(\alpha) \cap Z(\beta)$ has exactly two orbits of length 694,980 on $\Omega_2^3(\alpha)$, namely, $\Omega_2^3(\alpha) \cap \Omega_1(\beta)$ and $\Theta(\beta) := \Omega_2^3(\alpha) \cap \Omega_1(\beta^\alpha)$. Notice that if $\gamma \in \Theta(\beta)$ then $\beta\gamma$ is of order 3 and hence $\gamma \in \Omega_2^3(\beta)$. We claim that $\Gamma(Y_{333}, d_1)$ is isomorphic to the graph on $\Omega_2^3(\alpha)$ in which β is adjacent to $\Theta(\beta)$. In fact, the stabilizer in $Y_{333}/\langle f_{213}, f_{312}\rangle$ of a triangle in $\Gamma(Y_{333}, d_1)$ contains Sym_8 while if H is the stabilizer in B of a triangle in Ω then $H/O_2(H) \cong U_4(2)$ which does not involve Sym_8 and the claim follows.

Let Ξ be a graph on the set of vertices of Ω in which α and β are adjacent if $\beta \in \Omega_2^3(\alpha)$. By the above paragraph Ξ is weakly locally $\Gamma(Y_{333}, d_1)$ and checking the remaining conditions in Lemma 2.3 we conclude that $Z \cong 2 \cdot B$ is a Y_{433}-group. Notice that we realize the Coxeter generators of Y_{433} by involutions inside $2 \cdot B$. By Lemma 1.1 the direct product $2 \times 2 \cdot B$ is also a Y_{433}-group.

We claim that the homomorphism

$$\varphi\colon \bar{Y}_{433} := Y_{433}/\langle f_{213}, f_{312}\rangle \to B$$

induces a covering χ of Ω. Let $\tilde{P} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$ be a 2-path in $\Gamma(Y_{433}, e_1)$ which maps onto a 2-path $P = (s_1, s_2, s_3)$ in Ξ such that s_1 and s_3 are adjacent in Ω. Since φ induces a covering of $\Gamma(Y_{433}, e_1)$ onto Ξ, the stabilizer of $\tilde{P}$ in $\bar{Y}_{433}$ maps isomorphically onto the stabilizer of P in B and the latter is the edge stabilizer of the subgraph in Ω induced by $\Omega_2^3(\alpha)$, isomorphic to $2^6 : \mathrm{Sp}_6(2)$. We assume that $\tilde{s}_1 = Y_{433}\lfloor e_1 \rfloor$, so that $s_1 = B\lfloor e_1 \rfloor$. Let $\tilde{\Sigma}$ be the set of images of $\tilde{s}_1$ under $\bar{Y}_{433}\lfloor d_2 \rfloor$ and let Σ be the set of images of s_1 under $B\lfloor d_2 \rfloor$. Since

$$\bar{Y}_{433}\lfloor d_2 \rfloor \cong B\lfloor d_2 \rfloor \cong Y_{432}/\langle f_{213}\rangle \cong \mathrm{Fi}_{23},$$

$\tilde{\Sigma}$ maps bijectively onto Σ. Furthermore the action of $\bar{Y}_{433}\lfloor d_2 \rfloor$ on $\tilde{\Sigma}$ is similar to that of Fi_{23} on $\Delta(\mathrm{Fi}_{23})$. Assuming without loss of generality that $\tilde{P} \subset \tilde{\Sigma}$ we conclude from the suborbit diagram of $\Delta(\mathrm{Fi}_{23})$ that the stabilizer of $\{\tilde{s}_1, \tilde{s}_3\}$ in $\bar{Y}_{433}\lfloor d_2 \rfloor$ is of the form $2^2 \cdot U_6(2)$. Finally the stabilizer of $\{s_1, s_3\}$ in B, isomorphic to $2^{2+20}.U_6(2)$ is generated by its subgroups $2^6 : \mathrm{Sp}_6(2)$ and $2^2 \cdot U_6(2)$ which implies that φ induces a covering $\chi\colon \tilde{\Omega} \to \Omega$ of graphs. Here the vertices of $\tilde{\Omega}$ are the vertices of $\Gamma(Y_{433}, e_1)$ and the edges are the images of $\{\tilde{s}_1, \tilde{s}_3\}$ under Y_{433}. The subgraph in $\tilde{\Omega}$ induced by $\tilde{\Sigma}$ maps isomorphically onto the subgraph in Ω induced by Σ (both these subgraphs are isomorphic to $\Delta(\mathrm{Fi}_{23})$). Thus the triangles in Ω are contractible with respect to χ. It has been proved in [Iv92b] and [Iv94] (using the information on the antipodal folding of Ω deduced in [Seg91]) that Ω is triangulable. Hence both χ and φ are isomorphisms and $\bar{Y}_{433} \cong B$.

3.9. $Y_{443} \cong 2 \times M$

Let Z be the Monster group M and Λ be the Monster graph which is a graph on the conjugacy class of $2A$ (Baby Monster) involutions in the Monster with two involutions being adjacent if their product is again a $2A$ involution. If $\alpha \in \Lambda$ then $Z(\alpha) \cong 2 \cdot B$ (the commutator subgroup of Y_{433}) and the subgraph in Λ induced by $\Lambda_1(\alpha)$ is the graph Ω from the previous subsection together with a matching which joins pairs of antipodal vertices. This shows that Z has two orbits on the triangles in Λ. Every edge is contained in a unique triangle from one of the orbits (we call them short triangles) and in 3,968,055 triangles from another orbit (we call them long triangles). The suborbit diagram of Λ has been calculated in [Nor85] (see also [GMS89]) and we summarize the information on this subdiagram which we use in the following lemma (cf. [GMS89] for proofs).

LEMMA 3.8. *Let* $\alpha, \beta \in \Lambda$. *Then*

(i) *the orbit* $\Lambda^C(\alpha)$ *of* $Z(\alpha)$ *containing* β *is uniquely determined by the conjugacy class* C *which contains the product* $\alpha\beta$;

(ii) *the class* C *is one of the following*: $1A$, $2A$, $2B$, $3A$, $3C$, $4A$, $4B$, $5A$, $6A$ *and the corresponding 2-element stabilizers* $Z(\alpha) \cap Z(\beta)$ *are isomorphic to* $2 \cdot B$, $2^2 \cdot {}^2E_6(2)$, $2^{2+22}.\mathrm{Co}_2$, Fi_{23}, Th, $2^{1+22}.\mathrm{McL}$, $2.F_4(2)$, HN, $2 \cdot \mathrm{Fi}_{22}$, *respectively*;

(iii) *if* $\beta \in \Lambda^{3A}(\alpha)$ *then* $Z(\alpha) \cap Z(\beta)$ *acts transitively on the set of vertices* $\gamma \in \Lambda^{3A}(\alpha) \cap \Lambda^{2A}(\beta)$ *with stabilizer isomorphic to* $\mathrm{Sp}_8(2)$;

(iv) *if* $\beta \in \Lambda^{4B}(\alpha)$ *then* β *is at distance* 2 *from* α *in* Λ *and there is a unique* 2*-path joining these two vertices*;

(v) *if* $\beta \in \Lambda^{3A}(\alpha)$ *then the set* $\Lambda_1(\alpha) \cap \Lambda_1(\beta)$ *is of size* 31,671 *and it consists of the* $2A$ *involutions contained in* $Z(\alpha) \cap Z(\beta) \cong \mathrm{Fi}_{23}$.

Let Ξ be a graph on the vertex set of Λ in which α and β are adjacent if $\beta \in \Lambda^{3A}(\alpha)$. By Lemma 3.8(ii) the isomorphism

$$\sigma : \overline{Y}_{433} := Y_{433}/\langle f_{213}\rangle \to Z(\alpha)$$

induces an isomorphism of the permutation group $(\Gamma(Y_{433}, d_3), \overline{Y}_{433})$ onto the permutation group $(\Xi_1(\alpha), Z(\alpha))$. We denote the latter isomorphism by the same letter σ. We claim that whenever u and v are adjacent vertices in $\Gamma(Y_{433}, d_3)$, $\sigma(u)$ and $\sigma(v)$ are adjacent vertices in Ξ. First the stabilizer of $\{u, v\}$ in $\overline{Y}_{433}$ is isomorphic to $\mathrm{Sp}_8(2)$ and by Lemma 3.8(ii) we have $\sigma(u) \in \Lambda^C(\sigma(v))$ where C is $2A$, $3A$, or $4B$. Without loss of generality we assume that $u = Y_{433}\lfloor d_3 \rfloor$ and $v = Y_{433}\lfloor d_3 \rfloor d_3$. Then the isomorphism σ sends the Coxeter generators of $Y_{433}\lfloor d_3 \rfloor$ into $Z(\sigma(u))$ and by Lemma 3.8(v) the images of the generators are contained in $\Lambda_1(\alpha) \cap \Lambda_1(\sigma(u))$. The image under σ of d_3 maps $\sigma(u)$ onto $\sigma(v)$ and commutes with the images of the Coxeter generators of $Y_{433}\lfloor c_3, d_3 \rfloor$ which shows that $\sigma(u)$ and $\sigma(v)$ have at least nine common neighbours in Λ and by Lemma 3.8(iv) $\sigma(v) \notin \Lambda^{4B}(\sigma(u))$. Suppose that $\sigma(v) \in \Lambda^{2A}(\sigma(u))$. Then by Lemma 3.8(iii) $\Gamma(Y_{433}, d_3)$ maps isomorphically onto the subgraph in Λ induced by $\Xi_1(\alpha)$. We know from Subsection 3.5 that the stabilizer in $\overline{Y}_{433}$ of a triangle in $\Gamma(Y_{433}, d_3)$ is isomorphic to Sym_{10}. Clearly this must correspond to long triangles, but if H is the stabilizer in Z of a long triangle then $H/O_2(H) \cong U_6(2)$ and the latter group does not involve Sym_{10}. This contradiction shows that $\sigma(u)$ and $\sigma(v)$ are adjacent in Ξ. Hence Ξ is weakly locally $\Gamma(Y_{433}, d_3)$. It is easy to check the conditions in Lemma 2.3 and to conclude that $Z \cong M$ is a Y_{443}-group.

Next we claim that the homomorphism

$$\varphi\colon \overline{Y}_{443} := Y_{443}/\langle f_{312}\rangle \to M$$

induces a covering $\chi\colon \tilde{\Lambda} \to \Lambda$ of graphs with respect to which all long triangles are contractible. Consider a 2-path $\tilde{P} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$ in $\Gamma(Y_{443}, e_1)$ which maps onto a 2-path $P = (s_1, s_2, s_3)$ in Ξ such that s_1 and s_3 are adjacent in Λ. Then by Lemma 3.8(iii) and since φ induces a covering of $\Gamma(Y_{443}, e_1)$ onto Ξ, the stabilizer of $\tilde{P}$ in $\overline{Y}_{443}$ is isomorphic to $\mathrm{Sp}_8(2)$. Assume without loss of generality that $\tilde{s}_1 = \overline{Y}_{443}\lfloor e_1\rfloor$ and that $\tilde{s}_3$ is contained in the orbit of $\tilde{s}_1$ under $\overline{Y}_{443}\lfloor e_2\rfloor \cong 2\cdot B$. Then the suborbit diagram of Ω from the previous subsection shows that the stabilizer of $\{\tilde{s}_1, \tilde{s}_3\}$ in $Y_{443}\lfloor e_2\rfloor$ is isomorphic to $2^{3+20}.U_6(2)$. Finally since the stabilizer of $\{s_1, s_3\}$ in Z, isomorphic to $2^2\cdot{}^2E_6(2)$, is generated by its subgroups isomorphic to $2^{3+20}.U_6(2)$ and $\mathrm{Sp}_8(2)$ we conclude that φ indeed induces a covering $\chi\colon \tilde{\Lambda} \to \Lambda$ of graphs. The subgraph in $\tilde{\Lambda}$ induced by the images of $\tilde{s}_1$ under $\overline{Y}_{443}\lfloor e_2\rfloor$ is isomorphic either to the graph Ω from the previous subsection or to the subgraph in Λ induced by $\Lambda_1(\alpha)$ and in any case the long triangles are contractible with respect to χ. It was proved in [ASeg92] (using the information on the Monster graph deduced in [GMS89]) that Λ is triangulable. In [Iv94] using this result it was shown that the long triangle already generates the whole fundamental group of Ξ. Hence $\overline{Y}_{443} \cong M$ and in view of Lemma 1.1 we have $Y_{443} \cong 2 \times M$.

3.10. $Y_{444} \cong M \wr 2$

As above let M be the Monster group, let D be the direct product of two copies of M,

$$D = \{(g,h) \mid g,h \in M\}; \qquad (g_1,h_1)\cdot(g_2,h_2) = (g_1g_2, h_2h_1),$$

and define an action of D on M by $(g,h)\colon m \mapsto gmh$ for every $m \in M$ and $(g,h) \in D$. In this way we realize D as the group generated by the left and right regular representations of M. Let τ be the permutation on M acting by $\tau\colon m \mapsto m^{-1}$. Then τ can be considered as a permutation of D via $(g,h)^\tau = (h^{-1}, g^{-1})$, in particular τ normalizes D and permutes its direct factors. Define Z as the semidirect product of D and $\langle\tau\rangle$. Then it is easy to see that Z is the Bimonster. If $Z(1)$ is the stabilizer in Z of the identity element of M then $Z(1) = \langle\tau\rangle \times M'$ where $M' = \{(g,h) \in D \mid g = h^{-1}\}$ and hence every orbit of $Z(1)$ on M is of the form $C \cup C'$ where C is a conjugacy class of M and $C' = \{g^{-1} \mid g \in C\}$.

Let ξ be a graph on M in which m_1 and m_2 are adjacent if and only if $m_1m_2^{-1}$ is an element of type $3A$ in M. Since the class of $3A$ elements is closed under taking inverses, Z acts on Ξ vertex and edge transitively. If t

is of type $3A$ then the stabilizer in Z of the triple $T = \{1, t, t^{-1}\}$ is

$$\langle \tau \rangle \times N_{M'}(\langle t \rangle) \cong 2 \times 3 \cdot \mathrm{Fi}_{24},$$

which shows that the elementwise stabilizer in Z of the edge $\{1, t\}$ is isomorphic to $3 \cdot \mathrm{Fi}_{24}$ while the setwise stabilizer is of the form $3 \cdot \mathrm{Fi}_{24} \times 2$.

Let $A \cong \mathrm{Alt}_4$ be a subgroup in M with the monstralizer of the form $(\mathrm{Alt}_4 \times \Omega_{10}^-(2)) : 2$ (cf. [Nor98]). By the order reason all elements of order 3 in A are of type $3A$ and hence by choosing t_1 and t_2 to be suitable such elements, we obtain a triangle $T_1 = \{1, t_1, t_2\}$ in Ξ whose elementwise stabilizer is isomorphic to $\Omega_{10}^-(2) : 2$ (notice that T_1 is fixed by the product of an element in the monstralizer of A which inverts both t_1 and t_2 and the element τ). Hence Ξ is weakly locally $\Gamma(Y_{443}, d_3)$ and by checking the conditions in Lemma 2.3, we conclude that Z is a Y_{444}-group.

Let $\psi \colon \Gamma(Y_{444}, e_1) \to \Xi$ be the covering of graphs induced by the homomorphism of $Y_{444} \to Z$ and let Θ be a graph on M in which m_1 and m_2 are adjacent if $m_1 m_2^{-1}$ is an element of type $2B$ in M. We are going to show that ψ induces a covering χ of Θ and that certain triangles in Θ are contractible with respect to χ. Notice that the elementwise stabilizer in Z of an edge of Θ is isomorphic to $2 \times 2_+^{1+24}.\mathrm{Co}_1$. Consider in M a maximal 2-local subgroup

$$P \cong 2^{2+11+22}.(\mathrm{Sym}_3 \times \mathrm{Mat}_{24}),$$

and let s be an element of order 3 in $O_{2,3}(P)$. Then $C_P(s) \cong 2^{11}.(3 \times \mathrm{Mat}_{24})$, s is of type $3A$ by the order reason and $Z(O_2(P))$ is $2B$ pure of order 4. Hence for $s_1 \in sZ(O_2(P)) \setminus \{s\}$ we obtain a triple $T_2 = \{1, s, s_1\}$ such that $\{s, 1, s_1\}$ is a 2-arc in Ξ and $\{s, s_1\}$ is an edge in Θ. The elementwise stabilizer of T_2 in Z contains Mat_{24}. Since ψ is a covering, there is a pair of vertices $\{\tilde{s}, \tilde{s}_1\}$ in $\Gamma(Y_{444}, e_1)$ which maps onto an edge of Θ and whose stabilizer in Y_{444} contains Mat_{24}.

Put $H = Z\lfloor e_3 \rfloor$ and let Σ be the set of images under H of the identity element of M. Then by the previous subsection $H \cong Y_{444}\lfloor e_3 \rfloor \cong 2 \times M$. Furthermore if $\tilde{\Sigma}$ is the vertex-set of a connected component of the subgraph in $\Gamma(Y_{444}, e_1)$ induced by $\psi^{-1}(\Sigma)$ then the subgraph in $\Gamma(Y_{444}, e_3)$ induced by $\tilde{\Sigma}$ and the subgraph in Ξ induced by Σ are isomorphic to $\Gamma(Y_{443}, e_1)$ and H acts on Σ with kernel of order 2. Now without loss of generality in terms of the previous paragraph we can assume that $s, s_1 \in \Sigma$, $\tilde{s}, \tilde{s}_1 \in \tilde{\Sigma}$. Then the setwise stabilizer of $\{s, s_1\}$ in H (isomorphic to the stabilizer of $\{\tilde{s}, \tilde{s}_1\}$ in Y_{444}) is of the form $2 \times 2^{2+22}.\mathrm{Co}_2$. Since the stabilizer in Z of an edge in Θ is generated by any two of its subgroups isomorphic to Mat_{24} and to $2 \times 2^{2+22}.\mathrm{Co}_2$, we conclude that the stabilizer in Y_{444} of $\{\tilde{s}, \tilde{s}_1\}$ maps bijectively onto the stabilizer in Z of $\{s, s_1\}$ and hence ψ induced a covering $\chi \colon \tilde{\Theta} \to \Theta$ of graphs. Notice that the vertex-set of $\tilde{\Theta}$ is that of $\Gamma(Y_{444}, e_3)$ and the edges are the images under Y_{444} of the pair $\{\tilde{s}, \tilde{s}_1\}$.

It is clear that the covering χ induces an isomorphism of the subgraph in $\tilde{\Theta}$ induced by $\tilde{\Sigma}$ onto the subgraph in Θ induced by Σ. This means that every triangle in Σ is contractible with respect to χ. Such a triangle is formed for instance by the non-identity elements in $Z(O_2(P))$. Thus we conclude that whenever z_1, z_2, z_3 are elements of type $2B$ in M such that $z_1 z_2 z_3 = 1$ and $z_i \in O_2(C_M(z_j))$ for $1 \leq i, j \leq 3$, then the triangle in Θ induced by $\{1, z_1, z_2\}$ is contractible with respect to χ.

Now we are going to apply a result from [IPS96] to show that χ is an isomorphism. A direct factor M of D acts regularly on Θ and hence Θ can be considered as a Cayley graph of M so that the corresponding generators are the $2B$ involutions. Let

$$\delta : \hat{\Theta} \to \Theta,$$

the covering of Θ with respect to the subgroup in its fundamental group generated by the images under M of the triangles $\{1, z_1, z_2\}$ such that $z_1, z_2, z_3 := z_1 z_2$ are $2B$ involutions and $z_i \in O_2(C_M(z_j))$ for $1 \leq i, j \leq 3$. Let $\hat{M}$ be the group of all liftings of elements of M to autmorphisms of $\hat{\Theta}$. It is clear that the subgroup of deck transformations acts regularly on each fiber and hence $\hat{M}$ acts regularly on $\hat{\Theta}$. This means that $\tilde{\Theta}$ is a Cayley graph of $\hat{M}$ with respect to generators $t(z)$, one for every $2B$ involution z in M. Since $\hat{\Theta}$ is undirected the generators are involutions and since the triangle $\{1, z_1, z_2\}$ as above is contractible with respect to δ, the corresponding generators satisfy the equality $t(z_1)t(z_2)t(z_3) = 1$. The following result has been proved in [IPS96].

LEMMA 3.9. *Let $\hat{M}$ be a group generated by involutions $t(z)$, one for every $2B$-involution z in the Monster M such that $t(z_1)t(z_2)t(z_3) = 1$ whenever z_1, z_2, z_3 are $2B$ involutions in M such that $z_i \in O_2(C_M(z_j))$ for $1 \leq i, j \leq 3$ and $z_1 z_2 z_3 = 1$. Then $\hat{M} \cong M$.*

By Lemma 3.9 and the paragraph before it δ is an isomorphism. Hence χ is an isomorphism as well and $Y_{444} \cong M \wr 2$.

ACKNOWLEDGMENTS

I am very thankful to D. V. Pasechnik who has computed the suborbit diagrams of $\Gamma(Y_{422}, e_1)$, $\Gamma(Y_{322}, c_2)$, $\Gamma(Y_{332}, d_1)$ and to S. V. Shpectorov who has pointed out a possible connection between Y_{444} and the non-abelian representation of the tilde geometry of the Monster.

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