Title: <u>Another viewpoint of Euler graphs and Hamiltonian graphs</u>

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*Abstract:* It may appear that there is little left to do in regards to the study of the Hamiltonian property of vertex transitive graphs unless there is a major breakthrough on the famous Lovasz conjecture. However, if we extend the concept of the traditional Hamiltonian property to other Hamiltonicity properties, then there is still much left to explore. In this series of lectures, I will introduce some of these Hamiltonicity properties, namely fault tolerant Hamiltonian, spanning connectivity, and mutually independent Hamiltonicity.

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Mathematics is the study of topics such as quantity, structure, space, and change. Mathematicians seek out patterns and use them to formulate new conjectures. Mathematicians resolve the truth or falsity of conjectures by mathematical proof. When mathematical structures are good models of real phenomena, then mathematical reasoning can provide insight or predictions about nature. Through the use of abstraction and logic, mathematics developed from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Now, mathematics is used throughout the world as an essential tool in many fields, including natural science, engineering, medicine, finance and the social sciences.

However, the students are lost in the waste land of trouble and grief in the mathematics courses with of counting and calculation. Thus, I believe the curriculum in mathematics courses need to be flipped. For this reason, I try to write a book. The material of this book is introduced in the lecture. Euler graphs and Hamiltonian graphs are the main themes. Yet, my emphasis is on the abstraction ability, logic reasoning, creativity, and the attitude of facing difficult problems.

The puzzle of tracing a diagram with one stroke is introduced in the first lecture. After abstraction, graph models are built. In term of graph, an Eulerian trail is a path in a graph which visits every edge exactly once. Similarly, an Eulerian circuit or Eulerian cycle is an Eulerian trail which starts and ends on the same vertex. An Eulerian graph is a graph with Eulerian cycle. Tracing a diagram with one stroke corresponds to the abstract graph has an Euler cycle or an Eulerian trail. With the concept of odd numbers, even numbers, and connected, the puzzle of tracing a diagram with one stroke is solved. Moreover, the formal proof and the corresponding algorithm are realized.

Furthermore, we can create more puzzles concerning this topic. I believe that this material is suitable for primary school students.

In the first lecture, we discuss a cycle/path in a graph which visits every edge exactly once. In the second lecture we discuss a cycle in a graph which visits every vertex exactly once and in lecture 3, we discuss a path in a graph which visits every vertex exactly once. A cycle in a graph which visits every vertex exactly once is called a Hamiltonian cycle. A graph is Hamiltonian if it has a Hamiltonian cycle. In lecture 2, we introduce some interesting puzzles of Hamiltonian graphs. We note that there is a big difference between Eulerian graphs and Hamiltonian graphs. There exists a very efficient algorithm to recognize an Eulerian graph. However, to recognize a Hamiltonian graph is an NP-complete problem which means that it is very difficult to find an efficient algorithm to recognize a Hamiltonian graph. For this reason, we propose some brute force method to recognize a Hamiltonian graph. We also present some methods to determine some graphs are not Hamiltonian. This might be the most difficult part. However, I hope you can learn some logic reasoning. In other words, you are already in a waste land of trouble and grief.

A path in a graph which visits every vertex exactly once is called a Hamiltonian path. In lecture 3, we begin with some puzzles for Hamiltonian paths. Then, we introduce a concept of Hamiltonian connected. A Hamiltonian connected graph is a graph such that there is a Hamiltonian path between any two different vertices in the graphs. We note that the degree of any vertex in a Hamiltonian connected graph with more that vertices is at least 3. For this reason, we limit ourselves in cubic Hamiltonian connected graphs. A lot of cubic Hamiltonian connected graphs are proposed. We also interested in Hamiltonian path in certain bipartite graphs. A bipartite graph $G$ is a graph whose vertices can be divided into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$. Vertex set $U$ and $V$ are often denoted as *partite sets.* The cardinality of $U$ and $V$ are of the same and least 2 if $G$ is Hamiltonian. It is easy to see that there is no Hamiltonian path joining two vertices of the same partite set in any Hamiltonian bipartite graph. Yet, it is possible to find a Hamiltonian path joining two vertices of different partite sets. A Hamiltonian laceable graph if a bipartite graph such that there exists a Hamiltonian path between any two vertices of different partite sets. Again a lot of cubic Hamiltonian laceable graphs are proposed in this lecture.

It seem that the materials in lecture 3 are much difficult than the materials in lecture 2. Yet, the world is flipped. We are hard to improve anything if the material is easy. However, any tiny improvement is an improvement indeed if the material is hard.

Easy problems are flipped into hard problems and hard problems are flipped into easy problems. Thus, the attitude to face hard problems is important.

In the following lectures, I share my experience with the hard problems such as Hamiltonian cycles. Moreover, I will give suggested readings and possible directions at each lecture. It is easy to write possible directions as hard as possible because there are a lot of famous conjectures in Hamiltonian cycles. In this case, you will enter the waste land. However, I try to make the possible directions as easy as possible.

Actually, all the material in the first three lectures can be found in the old literatures. The story in lecture 4 and 5 begins with an open problem of Harary and Hayes. In 1993, Harary and Hayes wrote a paper “Edge fault tolerance in graphs” and in 1996, they wrote another paper “Node fault tolerance in graphs”. In the second paper, Harary and Hayes conjectured that all cubic 1-node fault tolerant Hamiltonian graphs are of certain family of graphs. I did not solve the conjecture. However, I found another paper by Mukhopadhyaya and Sinha in 1992. With their result, the conjecture of Harary and Hayes is disproved by finding another family of graphs. Then, we are lucky by find more and more families of graphs. In terms of diameter, we use the concept of diameter to show our result is better than previous result. Moreover, we compare the difference among cubic 1-edge fault tolerant Hamiltonian graphs, cubic 1-vertex fault tolerant Hamiltonian graphs, and Hamiltonian connected graphs. We also propose some construction schemes of finding such graphs. Moreover, I discuss the concept of globally 3*-connected graphs. A globally 3*-connected graph is a cubic graph such that there are 3 internal disjoint paths spanning all the vertices between any two vertices.

In lectures 6 and 7, I discuss the bipartite variant of the material in lectures 4 and 5. More precisely, I discuss the following material. A bipartite graph is 1p-fault tolerant Hamiltonian graph if it remains Hamiltonian if a pair of vertices, one from each partite set, is deleted. A bipartite graph is 1-edge fault tolerant Hamiltonian laceable if it remains Hamiltonian laceable if any edge is faulty. A strongly Hamiltonian laceble graph is Hamiltonian laceble graph such that there is a path visiting all vertices except one vertex between any two vertices of the same color. A hyper Hamiltonian laceble graph is a Hamiltonian laceble graph such that there is a fault free Hamiltonian path between any two vertices of the same color if one vertex of the different color is faulty. A globally Bi-3*-connected graph is a cubic bipartite graph such that there are three disjoint spanning paths between any two vertices of different color. A hyper globally Bi-3*-connected graph $G$ is a globally Bi-3*-connected graph such that for any three vertices $x, y, z$ of the same color.

In the final section, I discuss cubic 2-independent Hamiltonian connected graphs and 3-independent Hamiltonian graphs. Let $G$ be a Hamiltonian connected graph with $n$ vertices. Let $P_1 : u_1, u_2, \ldots, u_n$ and $P_2 : v_1, v_2, \ldots, v_n$ be any two Hamiltonian paths of $G$. We say that $P_1$ and $P_2$ are *independent* if $u_1 = v_1$, $u_n = v_n$, and $u_i \neq v_i$ for $1 < i < n$. A cubic graph $G$ is *2-independent Hamiltonian connected* if there are two independent Hamiltonian paths between any two different vertices of $G$. Let $G$ be a Hamiltonian graph with $n$ vertices. A hamiltonian cycle $C$ of a graph $G$ can be viewed as an ordered set $u_1, u_2, \ldots, u_n, u_1$ of vertices such that $u_i \neq u_j$ for $i \neq j$ and $u_i$ is adjacent to $u_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$ and $u_n$ is adjacent to $u_1$. The vertex $u_1$ is the starting vertex and $u_i$ is the $i$th vertex of $C$. Two hamiltonian cycles $C_1 : u_1, u_2, \ldots, u_n, u_1$ and $C_2 : v_1, v_2, \ldots, v_n, v_1$ of $G$ are *independent* if $u_1 = v_1$ and $u_i \neq v_i$ for every $i \in \{2, 3, \ldots, n\}$. A set of hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of $G$ is mutually independent if its elements are pairwise independent.