On finite minimal non-$\pi$-closed groups

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Let $G$ be a finite group and $\pi$ a set of primes. A group $G$ having a normal $\pi$-Hall subgroup is called $\pi$-closed. We consider groups $G$ such that $G$ is not $\pi$-closed and all maximal subgroups of $G$ are $\pi$-closed, i.e. $G$ is a minimal non-$\pi$-closed group.

Theorem 1 [1, Theorem 1']. If $G$ is a minimal non-$\pi$-closed group then either $G/\Phi(G)$ is a simple non-abelian group or $G$ is a Schmidt group.

Thus, the study of the minimal non-$\pi$-closed groups reduces practically to the case of the simple non-abelian groups. Further we use the following notation. As usual $\pi(n)$ is the set of all primes dividing a natural $n$. If $q$ is a prime power then $S(q) := \{ q_0 \in \mathbb{N} \mid q = q_0^r \text{ for some } r \}$. If $P(x)$ is a integral polynomial on $x$ then $\pi_P(P(q)) := \pi(P(q)) \setminus \bigcup_{q_0 \in S(q)} \pi(P(q_0))$.

Theorem 2 [2,3]. Let $G$ be a finite simple non-abelian group different from $PSL_r(q)$ and $PSU_r(q)$ with an odd prime $r$ and $E(q)$ (everywhere $q$ is a prime power), and $\pi \subseteq \pi(G)$. The following conditions are equivalent:

(A) $G$ is a minimal non-$\pi$-closed group;

(B) $2 \not\in \pi$, $\pi \neq \emptyset$ and one of the following conditions holds:

1. $G \cong A_r$ where $r \geq 5$ is a prime different from 11, 23 and $(q^n - 1)/(q - 1)$ where $q$ is a prime powers and $n \in \mathbb{N}$, and $\pi = \{ r \}$;

2. $G \cong PSL_2(q)$, $q > 5$, $\pi(q) = \{ p \}$, and one of the following conditions holds:

   (2a) $q = p$ and either $\pi \subseteq \pi(p + 1) \setminus \{ 3, 5 \}$ or $p \in \pi \subseteq \pi(p(p^2 - 1)) \setminus \{ 3, 5 \}$;

   (2b) $q = p^n > p$, $\pi \subseteq \pi_0(q + 1) \setminus \{ 5 \}$, and $3 \not\in \pi$ if $p > 2$;

3. $G \cong Sz(q)$ ($q = 2^{2n+1} \geq 8$, $\pi \subseteq \pi_0(q^2 + 1)$ for non-prime $2n + 1$ and $\pi \subseteq \pi(q^2 + 1)$ for prime $2n + 1$;

4. $G \cong 2G_2(q)$ ($q = 3^{2n+1} \geq 27$, $\pi \subseteq \pi_0(q^2 - q + 1)$ for non-prime $2n + 1$ and $\pi \subseteq \pi(q^2 - q + 1)$ for prime $2n + 1$;

5. $G \cong 3D_4(q)$ and $\pi \subseteq \pi_0(q^2 - q^2 + 1)$;

6. $G \cong 2G_2(q)$ ($q = 2^{2n+1} \geq 8$) and $\pi \subseteq \pi_0(q^4 - q^2 - q^2 + 1)$;

7. $G$ is one of the sporadic groups $M_{23}, J_1, J_4, L_9, F_{24}'$, $F_2$ and $\pi$ is as in [2, Theorem 2].

Thus, for the complete description of the all pairs $(G, \pi)$ where $G$ is a simple minimal non-$\pi$-closed group it remains to consider only three series of groups $G$: $PSL_r(q)$ and $PSU_r(q)$ with an odd prime $r$ and $E(q)$.

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References

[1] V. A. Belonogov, On finite groups whose all maximal subgroups are $\pi$-closed, Works of Int. Scool-Conf. on group theory, ded. to 70 of V. V. Kabanov, Nalchik, 2014, 6–9. (In Russian)
