On finite minimal non-π-closed groups

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Let G be a finite group and π a set of primes. A group G having a normal π-Hall subgroup is called π-closed. We consider groups G such that G is not π-closed and all maximal subgroups of G are π-closed, i.e. G is a minimal non-π-closed group.

Theorem 1 [1, Theorem 1′]. *If G is a minimal non-π-closed group then either $G/\Phi(G)$ is a simple non-abelian group or G is a Schmidt group.*

Thus, the study of the minimal non-π-closed groups reduces practically to the case of the simple non-abelian groups. Further we use the following notation. As usual $\pi(n)$ is the set of all primes dividing a natural n. If q is a prime power then $S(q) := \{q_0 \in \mathbb{N} \mid q = q_0^r$ for a some prime $r\}$. If $P(x)$ is a integral polynomial on x then $\pi_0(P(q)) := \pi(P(q)) \setminus \cup_{q_0 \in S(q)} \pi(P(q_0))$.

Theorem 2 [2,3]. *Let G be a finite simple non-abelian group different from $PSL_r(q)$ and $PSU_r(q)$ with an odd prime r and $E_8(q)$ (everywhere q is a prime power), and $\pi \subseteq \pi(G)$. The following conditions are equivalent:*

(A) *G is a minimal non-π-closed group;*

(B) *$2 \notin \pi$, $\pi \neq \varnothing$ and one of the following conditions holds:*

(1) *$G \cong A_r$ where $r \geq 5$ is a prime different from $11, 23$ and $(q^n - 1)/(q - 1)$ where q is a prime powers and $n \in \mathbb{N}$, and $\pi = \{r\}$;*

(2) *$G \cong PSL_2(q)$, $q > 5$, $\pi(q) = \{p\}$, and one of the following conditions holds:*

(2a) *$q = p$ and either $\pi \subseteq \pi(p+1) \setminus \{3, 5\}$ or $p \in \pi \subseteq \pi(p(p^2 - 1)) \setminus \{3, 5\}$;*

(2b) *$q = p^m > p$, $\pi \subseteq \pi_0(q + 1) \setminus \{5\}$, and $3 \notin \pi$ if $p > 2$;*

(3) *$G \cong Sz(q)$ $(q = 2^{2n+1} \geq 8)$, $\pi \subseteq \pi_0(q^2 + 1)$ for non-prime $2n + 1$ and $\pi \subseteq \pi(q^2 + 1)$ for prime $2n + 1$;*

(4) *$G \cong {}^2G_2(q)$ $(q = 3^{2n+1} \geq 27)$, $\pi \subseteq \pi_0(q^2 - q + 1)$ for non-prime $2n + 1$ and $\pi \subseteq \pi(q^2 - q + 1)$ for prime $2n + 1$;*

(5) *$G \cong {}^3D_4(q)$ and $\pi \subseteq \pi_0(q^4 - q^2 + 1)$;*

(6) *$G \cong {}^2F_4(q)$ $(q = 2^{2n+1} \geq 8)$ and $\pi \subseteq \pi_0(q^4 - q^2 + 1)$;*

(7) *G is one of the sporadic groups $M_{23}, J_1, J_4, Ly, Fi'_{24}$, F_2 and π is as in* [2, Theorem 2].

Thus, for the complete description of the all pairs (G, π) where G is a simple minimal non-π-closed group it remains to consider only three series of groups G: $PSL_r(q)$ and $PSU_r(q)$ with an odd prime r and $E_8(q)$.

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References

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