MDS codes with code distance at least 3 in Doob graphs.

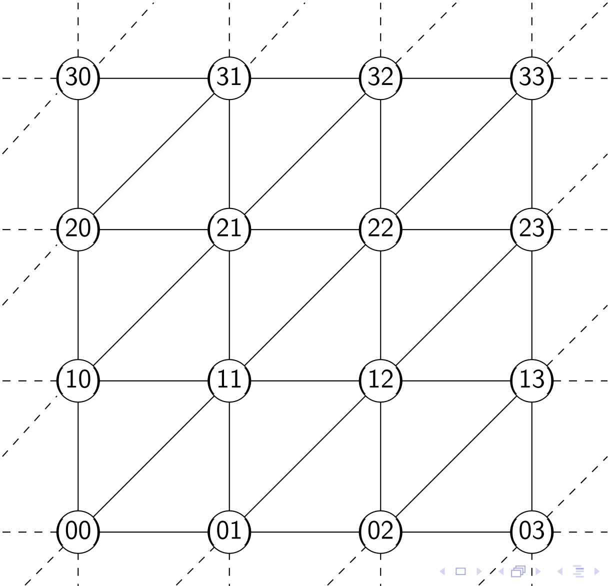
Evgeny Bespalov

Sobolev Institute of Mathematics

Shrikhande graph

- The Shrikhande graph was discovered in 1959 by S. S. Shrikhande.
- It is strongly regular graph with parametres $(v, k, \lambda, \mu) = (16, 6, 2, 2)$.
- The Shrikhande graph *Sh* can be considered as Cayley graph of Z_4^2 with the connecting set $\{01, 03, 10, 30, 11, 33\}$.

Shrikhande graph



The Doob graphs

- $D(m, n) = Sh^m \times K_4^n$
- If $m = 0$, then $D(0, n)$ is a Hamming graph $H(n, 4) = K_4^n$.
- If $m > 0$, then $D(m, n)$ is a Doob graph.
- $D(m, n)$ is a dictance-regular graph with the same parametres as $H(2m + n, 4)$.

MDS codes

- For any code C in $D(m, n)$ with code distance d, $|C| < 4^k$, $k = 2m + n - d + 1$.
- Code C in $D(m, n)$ with code distance d we call MDS code, if $|C| = 4^k$, $k = 2m + n - d + 1$.
- We denote such codes as $(m + n, 4^k, d)$ MDS codes, $d = 2m + n - k + 1$

MDS codes

- Two codes are said to be equivalent if there is a automorphism of Doob graph that maps one code to another
- $L_{m,n,k}$ — number of different $(m+n, 4^k, d = 2m+n-k+1)$ MDS codes up to the equivalence.

Main results

Theorem

- $L_{m,n,1} = m^3/36 + 7m^2/24 + 11m/12 + 1 - (m \mod 2)/8 - (m \mod 3)/9$;
- *if* $4 \le 2m + n \le 6$ *and* $3 \le d \le 4$, *then the values of* $L_{m,n,2m+n-d+1}$ *are shown in the table;*
- *if* $2m + n = 6$, *then* $L_{m,n,2} = 0$;
- *if* $2m + n > 6$ *and* $2 < d < 2m + n$, *then* $L_{m,n,2m+n-d+1} = 0$.

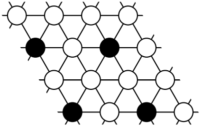
(m,n)	(2,0)	(1,2)	(2,1)	(1,3)	(2,2)	(1,4)	(3,0)
$d = 3$	2	1	2	1	0	0	0
$d = 4$	4	2	2	1	1	0	0

Table: Number of values $L_{m,n,2m+n-d+1}$

Lemma 1. *Let graph* $G_1(V, E_1)$ *be either Shrikhande graph or graph* K_4^2*. Ant let graph* $G_2(V, E_2)$ *be also either Srikhande graph or graph* K_4^2*, and* $E_1 \cap E_2 = \emptyset$*. Then* $G_3 = (V, E_3 = \overline{E_1 \cup E_2})$ *is the union of 4 disjoint graph* K_4*.*

Maximum independent sets

(a)



(b)

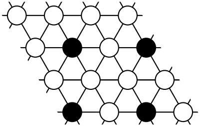


Figure: All maximum independence sets in *Sh* up to the equivalence

Partitions on the maximum independent sets

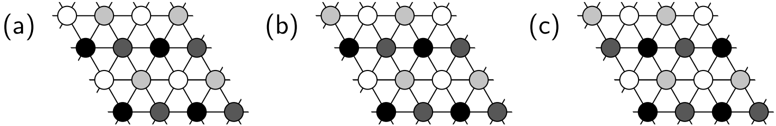


Figure: All partitions of graph *Sh* on the disjoint maximum independent sets up to the equivalence

MDS codes with parametres $(2+0, 4^2, 3)$ and $(1+2, 4^2, 3)$

Denote the vertices of $D(m, n)$ as $(s_1, \dots, s_m; h_1, \dots, h_n)$.
Let C is $(m+n, 4^k, d)$ MDS code.
If we fix some coordinates $(i_1, \dots, i_r; j_1, \dots, j_t)$ such that $2r + t = k$, then we can represent any other coordinate as a function of values in fix coordinates.

MDS codes with parametres $(2+0, 4^2, 3)$ and $(1+2, 4^2, 3)$

Vertices of $(2+0, 4^2, 3)$ MDS codes can be represented as $(a, f(a))$, where $a, f(a)$ are vertices of Sh.
From the code distance we have:

- f is a bijection;
- if $d(a,b) = 1$, then $d(f(a), f(b)) = 2$;
- if $d(f(a), f(b)) = 1$, then $d(a,b) = 2$.

MDS codes with parametres $(2+0, 4^2, 3)$ and $(1+2, 4^2, 3)$

Define graphs $G_1(V, E_1)$ and $G_2(V, E_2)$, where V is vertex set of Sh and

$$E_1 = \{(a_1, a_2) : d(a_1, a_2) = 1, a_1, a_2 \in V\},$$
$$E_2 = \{(a_1, a_2) : d(f(a_1), f(a_2)) = 1, a_1, a_2 \in V\}.$$

MDS codes with parametres $(2+1, 4^3, 3)$ and $(1+3, 4^3, 3)$

Let C is $(2+1, 4^3, 3)$ MDS code.
We can represent these vertices as $(f_k(a), a, k)$, $k \in \{0,1,2,3\}$, $a, f_k(a)$ are vertices in Sh.
$D_i = \{(f_i(a), a)$: a is the vertex in Sh $\}$

MDS codes with parametres $(2+1,4^3,3)$ and $(1+3,4^3,3)$

Lemma 2. *Let C be MDS codes with parametres $(2+1,4^3,3)$ or $(1+3,4^3,3)$. Then*

(i) *for any vertex a and any different $i,j \in \{0,1,2,3\}$ we have $d(f_i(a),f_j(a))=2$;*

(ii) *for any vertex a and any $i=1,2,3$:*

$$\{f_k(a): k=0,1,2,3\} = L^{D_0}(f_0(a)) = L^{D_i}(f_i(a));$$

(iii) *for any a and for any $i=1,2,3$:*

$$R^{D_i}(a) = R^{D_0}(a);$$

(iv) *for any $i=1,2,3$ and for any pair a and b:*

$$d(f_0(a),f_0(b)) = d(f_i(a),f_i(b)).$$

MDS codes with parametres $(2+1, 4^3, 3)$ and $(1+3, 4^3, 3)$

Lemma 3. *Let* $U = \{U_0, U_1, U_2, U_3\}$ *be a partition of Shrikhande graph on the disjoint maximum independent sets.*
Then there is unique set of automorphisms τ_1, τ_2, τ_3 *such that for any* $j = 0, 1, 2, 3$, *any* $i = 1, 2, 3$ *and any vertex* s *of Srikhande graph:*
1) if $s \in U_j$, *then* $\tau_i(s) \in U_j$*;*
2) $d(\tau_i(s), s) = 2$*;*
3) $d(\tau_i(s), \tau_j(s)) = 2$, $i \neq j$.

Main results

Theorem

- $L_{m,n,1} = m^3/36 + 7m^2/24 + 11m/12 + 1 - (m \mod 2)/8 - (m \mod 3)/9$.
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Table: Number of values $L_{m,n,2m+n-d+1}$

Thank you for your attention