MDS codes with code distance at least 3 in Doob graphs.

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The Shrikhande graph was discovered in 1959 by S. S. Shrikhande. It is a strongly regular graph with parameters 
\((v, k, \lambda, \mu) = (16, 6, 2, 2)\).

The Shrikhande graph \(Sh\) can be considered as a Cayley graph of \(Z_4^2\) with the connecting set \(\{01, 03, 10, 30, 11, 33\}\).
Shrikhande graph
The Doob graphs

- $D(m, n) = Sh^m \times K_4^n$
- If $m = 0$, then $D(0, n)$ is a Hamming graph $H(n, 4) = K_4^n$.
- If $m > 0$, then $D(m, n)$ is a Doob graph.
- $D(m, n)$ is a distance-regular graph with the same parameters as $H(2m + n, 4)$. 
MDS codes

- For any code $C$ in $D(m, n)$ with code distance $d$, $|C| < 4^k$, $k = 2m + n - d + 1$.
- Code $C$ in $D(m, n)$ with code distance $d$ we call MDS code, if $|C| = 4^k$, $k = 2m + n - d + 1$.
- We denote such codes as $(m + n, 4^k, d)$ MDS codes, $d = 2m + n - k + 1$
MDS codes

- Two codes are said to be equivalent if there is a automorphism of Doob graph that maps one code to another
- \( L_{m,n,k} \) — number of different \( (m + n, 4^k, d = 2m + n - k + 1) \) MDS codes up to the equivalence.
Main results

Theorem

- \( L_{m,n,1} = \frac{m^3}{36} + \frac{7m^2}{24} + \frac{11m}{12} + 1 - \left( m \mod 2 \right)/8 - \left( m \mod 3 \right)/9 \);

- if \( 4 \leq 2m + n \leq 6 \) and \( 3 \leq d \leq 4 \), then the values of \( L_{m,n,2m+n-d+1} \) are shown in the table;

- if \( 2m + n = 6 \), then \( L_{m,n,2} = 0 \);

- if \( 2m + n > 6 \) and \( 2 < d < 2m + n \), then \( L_{m,n,2m+n-d+1} = 0 \).

<table>
<thead>
<tr>
<th>((m, n))</th>
<th>(2,0)</th>
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<tbody>
<tr>
<td>( d = 3 )</td>
<td>2</td>
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Table: Number of values \( L_{m,n,2m+n-d+1} \)
Lemma 1. Let graph $G_1(V, E_1)$ be either Shrikhande graph or graph $K_4^2$. And let graph $G_2(V, E_2)$ be also either Srikhande graph or graph $K_4^2$, and $E_1 \cap E_2 = \emptyset$. Then $G_3 = (V, E_3 = E_1 \cup E_2)$ is the union of 4 disjoint graph $K_4$. 
Maximum independent sets

Figure: All maximum independence sets in $Sh$ up to the equivalence
Partitions on the maximum independent sets

Figure: All partitions of graph $Sh$ on the disjoint maximum independent sets up to the equivalence
MDS codes with parameters \((2 + 0, 4^2, 3)\) and \((1 + 2, 4^2, 3)\)

Denote the vertices of \(D(m, n)\) as \((s_1, \ldots, s_m; h_1, \ldots, h_n)\).
Let \(C\) be a \((m + n, 4^k, d)\) MDS code.
If we fix some coordinates \((i_1, \ldots, i_r; j_1, \ldots, j_t)\) such that \(2r + t = k\),
then we can represent any other coordinate as a function of values in fixed coordinates.
MDS codes with parameters \((2 + 0, 4^2, 3)\) and \((1 + 2, 4^2, 3)\)

Vertices of \((2 + 0, 4^2, 3)\) MDS codes can be represented as \((a, f(a))\), where \(a, f(a)\) are vertices of \(Sh\).

From the code distance we have:

- \(f\) is a bijection;
- if \(d(a, b) = 1\), then \(d(f(a), f(b)) = 2\);
- if \(d(f(a), f(b)) = 1\), then \(d(a, b) = 2\).
MDS codes with parameters \((2 + 0, 4^2, 3)\) and \((1 + 2, 4^2, 3)\)

Define graphs \(G_1(V, E_1)\) and \(G_2(V, E_2)\), where \(V\) is the vertex set of Sh and

\[
E_1 = \{(a_1, a_2) : d(a_1, a_2) = 1, a_1, a_2 \in V\},
\]

\[
E_2 = \{(a_1, a_2) : d(f(a_1), f(a_2)) = 1, a_1, a_2 \in V\}.
\]
Let $C$ is $(2 + 1, 4^3, 3)$ MDS code.
We can represent these vertices as $(f_k(a), a, k)$, $k \in \{0, 1, 2, 3\}$, $a, f_k(a)$ are vertices in $Sh$.
$D_i = \{(f_i(a), a) : a$ is the vertex in $Sh$ $\}$
Lemma 2. Let $C$ be MDS codes with parametres $(2 + 1, 4^3, 3)$ or $(1 + 3, 4^3, 3)$. Then

(i) for any vertex $a$ and any different $i, j \in \{0, 1, 2, 3\}$ we have $d(f_i(a), f_j(a)) = 2$;

(ii) for any vertex $a$ and any $i = 1, 2, 3$:

$$\{f_k(a) : k = 0, 1, 2, 3\} = L^{D_0}(f_0(a)) = L^{D_i}(f_i(a));$$

(iii) for any $a$ and for any $i = 1, 2, 3$:

$$R^{D_i}(a) = R^{D_0}(a);$$

(iv) for any $i = 1, 2, 3$ and for any pair $a$ and $b$:

$$d(f_0(a), f_0(b)) = d(f_i(a), f_i(b)).$$
Lemma 3. Let $U = \{U_0, U_1, U_2, U_3\}$ be a partition of Shrikhande graph on the disjoint maximum independent sets. Then there is unique set of automorphisms $\tau_1, \tau_2, \tau_3$ such that for any \(j = 0, 1, 2, 3\), any \(i = 1, 2, 3\) and any vertex \(s\) of Shrikhande graph:
1) if \(s \in U_j\), then $\tau_i(s) \in U_j$;
2) $d(\tau_i(s), s) = 2$;
3) $d(\tau_i(s), \tau_j(s)) = 2, i \neq j$. 
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Thank you for your attention