Automorphism groups of cyclotomic schemes over finite near-fields

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An algebraic structure $\mathbb{K} = \langle \mathbb{K}, +, \circ \rangle$ is called a (right) *near-field*, if $\mathbb{K}^+ = \langle \mathbb{K}, + \rangle$ is a group, $\mathbb{K}^\times = \langle \mathbb{K}\backslash\{0\}, \circ \rangle$ is a group, $(x + y) \circ z = x \circ z + y \circ z$ for all $x, y, z \in \mathbb{K}$, and $x \circ 0 = 0$ for all $x \in \mathbb{K}$. Finite near-fields can be constructed via finite fields except for a finite number of near-fields [1]. The first ones are called *Dickson* near-fields, the last ones are *Zassenhaus* near-fields.

Let $\mathbb{K}$ be a finite near-field and K be a subgroup of the group $\mathbb{K}^\times$, $\mathcal{R}_K = \{R_K(a) \mid a \in \mathbb{K}\}$, where $R_K(a) = \{(x, y) \in \mathbb{K}^2 \mid x - y \in K \circ a\}$. The pair $(\mathbb{K}, \mathcal{R}_K)$ is called *cyclotomic scheme* over the near-field $\mathbb{K}$ with the base group K. Cyclotomic schemes over finite fields were defined by Delsarte for the algebraic theory of codes [2], cyclotomic schemes over finite near-fields were introduced in [4].

The *automorphism group* of the cyclotomic scheme $\mathcal{C} = (\mathbb{K}, \mathcal{R}_K)$ can be defined as the automorphism group of its partition $\mathcal{R}_K$, namely, $\mathrm{Aut}(\mathcal{C}) = \{g \in \mathrm{Sym}(\mathbb{K}) \mid R^g = R, R \in \mathcal{R}_K\}$. Observe that $\mathrm{Aut}(\mathcal{C}) = \mathrm{Sym}(\mathbb{K})$ if the base group of the cyclotomic scheme $\mathcal{C}$ equals $\mathbb{K}^\times$.

If $\mathbb{F}$ is the finite field of order q, K is a proper subgroup of $\mathbb{F}^\times$, then the automorphism group of the cyclotomic scheme $\mathcal{C} = (\mathbb{K}, \mathcal{R}_K)$ is a subgroup of $\mathrm{A\Gamma L}(1, q) = \{x \mapsto x^\sigma b + c \mid x \in \mathbb{F}, b \in \mathbb{F}^\times, c \in \mathbb{F}^+, \sigma \in \mathrm{Aut}(\mathbb{F})\}$ [3]. The same result was achieved in [4] for cyclotomic schemes over Dickson near-fields with some restrictions on the orders of their base groups. Here we complete the description of the automorphism groups of cyclotomic schemes over finite near-fields.

Theorem. *Let $\mathbb{K}$ be a finite near-field of order q, K a proper subgroup of $\mathbb{K}^\times$, and $\mathcal{C} = (\mathbb{K}, \mathcal{R}_K)$ the corresponding cyclotomic scheme. Then $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, q)$ except for a finite number of exceptional schemes. If $\mathcal{C}$ is one of the exceptions, then the subgroup H of $\mathrm{Sym}(\mathbb{K})$ with $\mathrm{Aut}(\mathcal{C}) \leq H$ is determined. In particular, if the base group K is solvable, then so is $\mathrm{Aut}(\mathcal{C})$.*

It is worth mentioning that one of the key tools of our proof is the recent classification of $\frac{3}{2}$-transitive permutation groups [5].

References

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