Automorphism groups of cyclotomic schemes over finite near-fields

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(based on joint paper with Andrey Vasil'ev)

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Near-fields

An algebraic structure $\mathbb{K} = \langle \mathbb{K}, +, \circ \rangle$ is called a (right) near-field, if

- $\mathbb{K}^+ = \langle \mathbb{K}, + \rangle$ is a group (with neutral element 0)
- $\mathbb{K}^\times = \langle \mathbb{K} \setminus \{0\}, \circ \rangle$ is a group
- $(x + y) \circ z = x \circ z + y \circ z, \; x, y, z \in \mathbb{K}$
- $x \circ 0 = 0, \; x \in \mathbb{K}$.

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If $\mathbb{K}$ is a finite near-field, then $\mathbb{K}^+ \simeq \mathbb{Z}_p^k$.

Classification of finite near-fields (Zassenhaus, 1936)

Every finite near-field is one of the following:

1. Dickson near-fields (constructed via finite fields),
2. Zassenhaus near-fields (7 exceptional near-fields).

Cyclotomic schemes over finite near-fields

Let $\mathbb{K}$ be a finite near-field, $H \leq \mathbb{K}^{\times}$.
For $a \in \mathbb{K}$ define

$$R_H(a) = \{(x, y) \in \mathbb{K}^2 \mid y - x \in H \circ a\}.$$

Set $\mathcal{R}_H = \{R_H(a) \mid a \in \mathbb{K}\}$ is a partition of $\mathbb{K}^2$.

The pair $\langle \mathbb{K}, \mathcal{R}_H \rangle$ is called the cyclotomic scheme $\mathcal{C} = \mathcal{C}(\mathbb{K}, H)$ over the near-field $\mathbb{K}$ with the base group H.

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$\mathrm{Aut}(\mathcal{C}) = \{g \in \mathrm{Sym}(\mathbb{K}) \mid R^g = R, R \in \mathcal{R}_H\}$ is an automorphism group of the scheme $\mathcal{C}$.

Main problem

Find automorphism groups of cyclotomic schemes over finite near-fields.

2-closure of permutation groups

Let G be a permutation group on Ω.

Action of G on Ω induces the action on Ω^2: $(\alpha, \beta)^g = (\alpha^g, \beta^g)$

Denote as $\mathrm{Orb}_2(G)$ the set of orbits of its action (2-orbits).

$G^{(2)} = \mathrm{Aut}(\mathrm{Orb}_2(G)) = \{g \in \mathrm{Sym}(\Omega) : O^g = O, O \in \mathrm{Orb}_2(G)\}$ is the 2-closure of G.

Note that $G \leq G^{(2)}$.

2-closure problem

Given a permutation gorup G, find a set of generators of $G^{(2)}$.

Let $\mathbb{K}$ be a finite near-field, $H \leq \mathbb{K}^\times$, $\mathcal{C} = \langle \mathbb{K}, \mathcal{R}_H \rangle$.

$G = G(\mathbb{K}, H) := \{x \mapsto x \circ b + c \mid x \in \mathbb{K}, b \in H, c \in \mathbb{K}^+\} \simeq \mathbb{K}^+ \rtimes H$
is the cyclotomic group over $\mathbb{K}$ with the base group H.

Note that $\mathrm{Orb}_2(G) = \mathcal{R}_H$, so $G^{(2)} = \mathrm{Aut}(\mathcal{C})$.

In particular, $G \leq \mathrm{Aut}(\mathcal{C})$.

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Scheme $\mathcal{C}(\mathbb{K}, H)$ is trivial if $H = \mathbb{K}^\times$.
For a trivial scheme $\mathcal{C}(\mathbb{K}, \mathbb{K}^\times)$, $\mathrm{Aut}(\mathcal{C}) = \mathrm{Sym}(\mathbb{K})$.

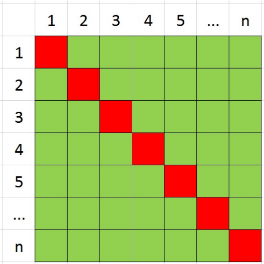

Figure: trivial scheme $\mathcal{C}(\mathbb{K}, \mathbb{K}^\times)$

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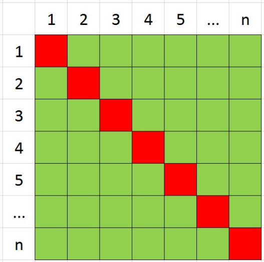

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Scheme $\mathcal{C}(\mathbb{K}, H)$ is nontrivial, if $H < \mathbb{K}^\times$.

P. Delsarte. An Algebraic Approach to the Association Schemes of Coding Theory, 1973:

$\mathcal{C} = \mathcal{C}(\mathbb{F}, H)$, where $\mathbb{F}$ is a finite field, $H \leq \mathbb{F}^{\times}$.
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Theorem (corollary from McConnel's work, 1963)

Let $\mathcal{C} = \mathcal{C}(\mathbb{F}, H)$ be a nontrivial cyclotomic scheme over a finite field $\mathbb{F}$ of order q with basis group H. Then $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, q) = \{x \mapsto x^{\sigma} \cdot b + c \mid x \in \mathbb{F}, b \in \mathbb{F}^{\times}, c \in \mathbb{F}^{+}, \sigma \in \mathrm{Aut}(\mathbb{F})\}$.

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Bagherian, Ponomarenko, Rahnamai Barghi, 2008:

- Cyclotomic scheme over finite near-field,
- $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, q)$ for some cyclotomic schemes over Dickson near-fields,
- conjecture: $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, q)$ for all finite near-fields, except for a finite number of near-fields.

Dickson near-fields

Finite near-field $\mathbb{K}$ is called Dickson near-field, if

- exists field $\mathbb{F}_0$ of order p^d and its extension $\mathbb{F}$ of degree n such that $\mathbb{F}^+ = \mathbb{K}^+$,
- $y \circ x = y^{\sigma_x} \cdot x, \quad x, y \in \mathbb{K}, \sigma_x \in \mathrm{Aut}(\mathbb{F}/\mathbb{F}_0)$.

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Obviously, $|\mathbb{K}| = p^{dn}$. For triple $\langle p, d, n \rangle$ exists Dickson near-field of order p^{dn}, if

$$\forall r \in \pi(n) \; r | p^d - 1 \;\; 4|n \Rightarrow 4|p^d - 1.$$

There are nonisomorphic near-fields of the same order.

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Theorem (Bagherian, Ponomarenko, Rahnamai Barghi, 2008)

Let $\mathbb{K}$ be a Dickson near-field of order p^{dn}, and $\mathcal{C} = \mathcal{C}(\mathbb{K}, H)$, such that $|H|$ has sufficiently large primitive Zsigmondy's divisor of pair (p, dn). Then $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, p^{dn})$.

Prime r is a primitive Zsigmondy's divisor of pair (p, dn), if $r | p^{dn} - 1$ and $r \nmid p^i - 1$ for $i < dn$.

Properties of $G(\mathbb{K}, H)$

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Permutation group on Ω is $\frac{3}{2}$-transitive, if it is transitive, and stabilizer of point α is $\frac{1}{2}$-transitive on $\Omega \setminus \{\alpha\}$.

Lemma

Let $\mathbb{K}$ be a finite near-field, $H < \mathbb{K}^\times$, $G = G(\mathbb{K}, H)$.
Then both G and $G^{(2)}$ are $\frac{3}{2}$-transitive.

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Liebeck, Praeger, Saxl, 2015

The classification of $\frac{3}{2}$-transitive permutation groups and $\frac{1}{2}$-transitive linear groups.

Dickson near-field case

Lemma (Bagherian, Ponomarenko, Rahnamai Barghi, 2008)

Let $\mathbb{K}$ be a Dickson near-field, $H < \mathbb{K}^{\times}$, $G = G(\mathbb{K}, H)$. Then $G^{(2)}$ is $\frac{3}{2}$-transitive group of affine type, i. e. $G^{(2)} = \mathbb{K}^{+} \rtimes L, H \leq L$.

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Classification of $\frac{3}{2}$-transitive permutation groups

Let X be a $\frac{3}{2}$-transitive permutation group. Then one of the following holds.

1. X is 2-transitive.
2. X is a Frobenius group.
3. X is almost simple.
4. X is affine, $X = N \rtimes L \leq \mathrm{AGL}(V)$, where $L \leq \mathrm{GL}(V)$, and L is a $\frac{1}{2}$-transitive linear group.

Then $G^{(2)} = \mathbb{K}^{+} \rtimes L$, L is a $\frac{1}{2}$-transitive linear group.

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Classification of $\frac{1}{2}$-transitive linear groups

If $L \leq \mathrm{GL}(V) = \mathrm{GL}(d, p)$ is $\frac{1}{2}$-transitive on $V^{\sharp} = V \setminus \{\bar{0}\}$, then one of the following holds:

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2. $L \leq \Gamma\mathrm{L}(1, p^d)$. $\Rightarrow G^{(2)} \leq \mathrm{A}\Gamma\mathrm{L}(1, q)$. ✓
3. L is a Frobenius complement acting semiregularly on $V^\sharp$.

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4. $L \leq \mathrm{GL}(2, p^{\frac{d}{2}})$, L is a group of all monomial matrices of determinant ± 1. ⊗
5. L solvable and $p^d = 3^2, 5^2, 7^2, 11^2, 17^2$ or 3^4. (GAP, MAGMA)

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6. $\mathrm{SL}(2, 5) \lhd L \leq \Gamma\mathrm{L}(2, p^{\frac{d}{2}})$, where $p^{\frac{d}{2}} = 9, 11, 19, 29$ or 169. ⊗

Main theorem

$\mathbb{K}$ is a finite near-field of order q, $\mathcal{C} = \mathcal{C}(\mathbb{K}, H)$ is a nontrivial cyclotomic scheme over $\mathbb{K}$ with base group H. Then one of the following statements hold.

1. $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, q)$.

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2. $\mathbb{K}$ is a Dickson near-field of order 7^2, $H \simeq Q_8 \times \mathbb{Z}_3$ and $\mathrm{Aut}(\mathcal{C}) = \mathbb{Z}_7^2 \rtimes (\mathrm{SL}(2,3) \times \mathbb{Z}_3)$.

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3. $\mathbb{K}$ is a Zassenhaus near-field, H is a solvable subgroup of $\mathbb{K}^\times$, $\mathrm{Aut}(\mathcal{C}) \leq \mathbb{K}^+ \rtimes L$, where $H \leq L$, and L is a known solvable group.

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2. $\mathbb{K}$ is a Dickson near-field of order 7^2, $H \simeq Q_8 \times \mathbb{Z}_3$ and $\mathrm{Aut}(\mathcal{C}) = \mathbb{Z}_7^2 \rtimes (\mathrm{SL}(2,3) \times \mathbb{Z}_3)$.
3. $\mathbb{K}$ is a Zassenhaus near-field, H is a solvable subgroup of $\mathbb{K}^\times$, $\mathrm{Aut}(\mathcal{C}) \leq \mathbb{K}^+ \rtimes L$, where $H \leq L$, and L is a known solvable group.
4. $\mathbb{K}$ is a Zassenhaus near-field of order either 29^2 or 59^2, $H \simeq \mathrm{SL}(2,5)$, and either $\mathrm{Aut}(\mathcal{C}) = \mathbb{Z}_{29}^2 \rtimes (\mathrm{SL}(2,5) \rtimes \mathbb{Z}_2)$ or $\mathrm{Aut}(\mathcal{C}) = \mathbb{Z}_{59}^2 \rtimes \mathrm{SL}(2,5)$.

Main theorem

$\mathbb{K}$ is a finite near-field of order q, $\mathcal{C} = \mathcal{C}(\mathbb{K}, H)$ is a nontrivial cyclotomic scheme over $\mathbb{K}$ with base group H. Then one of the following statements hold.

1. $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, q)$.
2. $\mathbb{K}$ is a Dickson near-field of order 7^2, $H \simeq Q_8 \times \mathbb{Z}_3$ and $\mathrm{Aut}(\mathcal{C}) = \mathbb{Z}_7^2 \rtimes (\mathrm{SL}(2,3) \times \mathbb{Z}_3)$.
3. $\mathbb{K}$ is a Zassenhaus near-field, H is a solvable subgroup of $\mathbb{K}^\times$, $\mathrm{Aut}(\mathcal{C}) \leq \mathbb{K}^+ \rtimes L$, where $H \leq L$, and L is a known solvable group.
4. $\mathbb{K}$ is a Zassenhaus near-field of order either 29^2 or 59^2, $H \simeq \mathrm{SL}(2,5)$, and either $\mathrm{Aut}(\mathcal{C}) = \mathbb{Z}_{29}^2 \rtimes (\mathrm{SL}(2,5) \rtimes \mathbb{Z}_2)$ or $\mathrm{Aut}(\mathcal{C}) = \mathbb{Z}_{59}^2 \rtimes \mathrm{SL}(2,5)$.

In particular, if H is solvable, then so is $\mathrm{Aut}(\mathcal{C})$.