On even-closedness of vertex-transitive graphs

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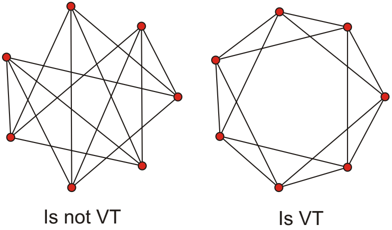
Overview

- Vertex-transitive graphs from pairs of groups
- Determining the full automorphism group
- Symmetry through odd automorphisms (STOA): Core research problem
- Cubic symmetric graphs

- An automorphism of a graph $X = (V, E)$ is an isomorphism of X with itself. Thus each automorphism α of X is a permutation of the vertex set V which preserves adjacency.
- A graph X is vertex-transitive if its automorphism group $Aut(X)$ acts transitively on vertices.



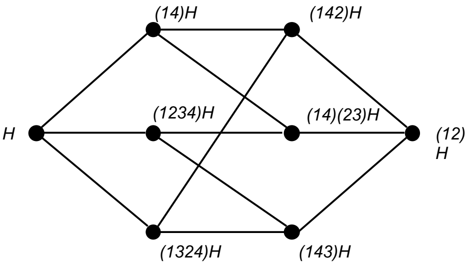
Vertex-transitive graphs - alternative (constructive) definition

Let (G, H) be a pair of abstract groups such that $H \leq G$,

- G/H ... the set of left cosets of G with respect to H.
- G acts on G/H by left multiplication as a transitive permutation group.
- This action induces an action of G on $G/H \times G/H$; the corresponding orbits are called orbitals.
- Define a graph with vertex set G/H and the edge set $E = Clo(\mathcal{O})$, where $\mathcal{O}$ is a union of orbitals and $Clo(\mathcal{O})$ a symmetric closure of $\mathcal{O}$, that is, if $(x, y) \in \mathcal{O}$ then both (x, y) and (y, x) are in $Clo(\mathcal{O})$.
- This graph is vertex-transitive, and every vertex-transitive graph can be obtained in this way.

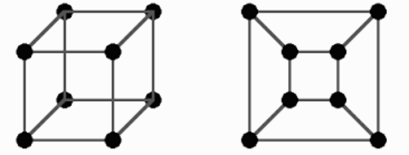
Example - the cube Q_3

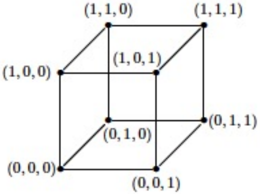
$G = S_4$
$H = \langle (1\,2\,3) \rangle \cong \mathbb{Z}_3$
$V = S_4/\mathbb{Z}_3$
$E = \mathcal{O}$, where $\mathcal{O}$ is the orbital containing the pair $(H, (1\,4)H)$.



The cube Q_3 - Alternatively

The cube can also be obtained from pair $G = \mathbb{Z}_2^3$ and $H = 1$.





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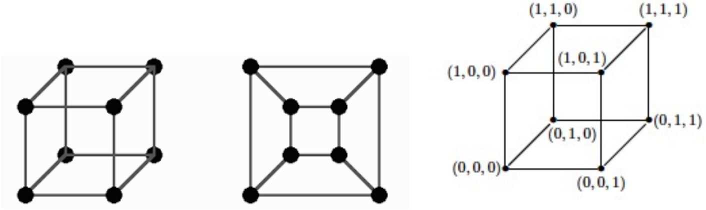

In general, we say that a vertex-transitive graph $X = Cay(G,S)$ is a Cayley graph if it can be obtained from a pair $(G,1)$ where $S = S^{-1} \subseteq G \setminus \{1\}$ denotes the set of neighbors of 1 in X.

$$Q_3 = Cay(\mathbb{Z}_2^3, \{100, 010, 001\})$$

The full automorphism group of a vertex-transitive graph

If X is a vertex-transitive graph arising from a pair (G, H) then $G \leq Aut(X)$.

What is $Aut(X)$, i.e. when is $G = Aut(X)$?

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Back to the cube Q_3:

Clearly, $\mathbb{Z}_2^3 \leq Aut(Q_3)$ and $S_4 \leq Aut(Q_3)$.

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$$Aut(Q_3) \cong S_4 \times \mathbb{Z}_2.$$

A crucial question in algebraic graph theory and beyond:

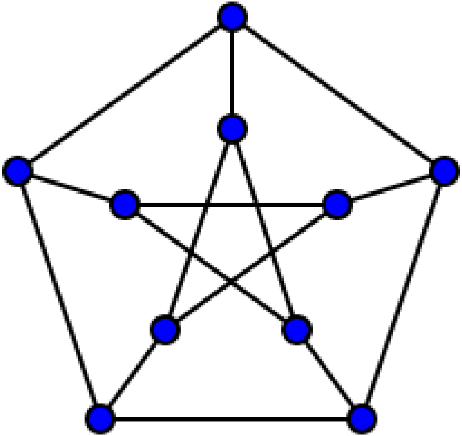
Given a graph, are there any symmetries beyond the obvious ones, and, if yes, how can one determine the full set?

We approach this question by building on the duality of even/odd permutations associated with graphs.

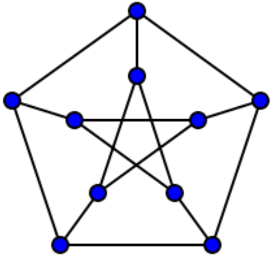
Even/odd automorphisms

An automorphism of a graph is said to be even/odd if it acts on the set of vertices as an even/odd permutation.

The Petersen graph



The Petersen graph



Obvious automorphisms:

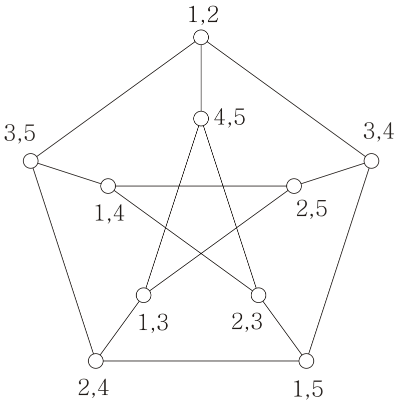
- rotation
- reflection

Semi-obvious automorphism:

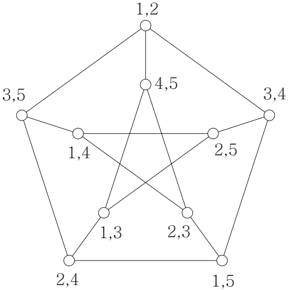
- swap (inner/outer 5-cycle)

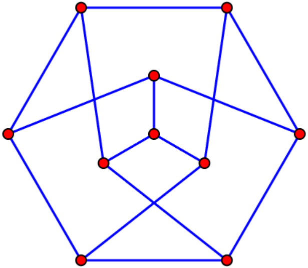
There exist additional automorphisms (mixers), disrespecting inner and outer cycles.

The Petersen graph



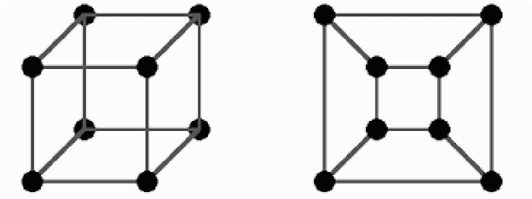
$Aut(X) \cong S_5$





The full automorphism group of the Petersen graph contains involutions with three orbits of size 2 and four fixed vertices, and hence odd (as permutations) automorphisms.

The cube Q_3



All automorphisms of the cube are even.

2-closure $H^{(2)}$ of H $=$ intersection of automorphism groups of all basic orbital graphs of H (basic $=$ arising from single orbitals)

H is 2-closed if $H^{(2)} = H$.

Even group $=$ group with only even permutations.
Odd group $=$ group containing also odd permutations.

Question

For H even, can we imbed it into an odd group via basic orbital graphs?

- If H is not 2-closed, is the 2-closure $H^{(2)}$ odd?
- If $H^{(2)}$ even, is there at least one basic orbital graph of H admitting an odd automorphism? If yes H is orbital-odd, otherwise H is even-closed.

Back to the cube and the Petersen graph:

The Petersen graph:
For $H = A_5$ the 2-closure is S_5, and is odd.
So $H = A_5$ as a group of degree 10 is orbital-odd.

The cube:
For $H \in \{\mathbb{Z}_2^3, S_4\}$ the 2-closure is $S_4 \times \mathbb{Z}_2$, and is even.
Still, $H \in \{\mathbb{Z}_2^3, S_4\}$ as a group of degree 8 is orbital-odd.

F102 – a cubic symmetric graph of type $\{4^1\}$:
Its automorphism group is isomorphic to $\mathrm{PSL}(2, 17)$.
Magma calculations show that $\mathrm{PSL}(2, 17)$ as a group of degree 102 is even-closed.

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For some integers n, all transitive groups of degree n are orbital-odd.

For example, $2p$, where p is a prime, is such an integer.

Odd automorphisms in VT graphs of order $2p$, p prime

X VTG of order $2p$, $G \leq \mathrm{Aut} X$ transitive

- G is imprimitive, blocks of size p;
- G is imprimitive, blocks of size 2,
- G is primitive.

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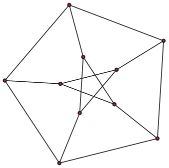
All swaps are odd automorphisms!

G is primitive

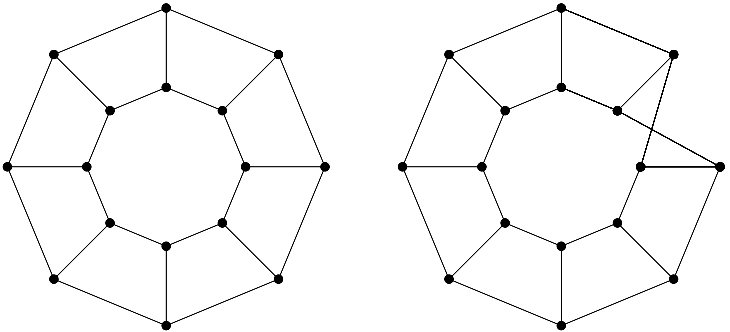
Example:
$p = 5$, A_5, S_5 acting on pairs from $\{1, 2, 3, 4, 5\}$.
Associated graphs: the Petersen graph and its complement.

By CFSG a primitive group of degree $2p$, $p > 5$, is double transitive, and as such the corresponding orbital graph is the complete graph K_{2p}.

No CFSG-free proof of this fact exists.



Also, no CFSG-free answer to
the even/odd question for VTG of order $2p$ exists.



Two examples of cubic vertex-transitive graphs, one with and one without odd automorphisms.

Proposition

A Cayley graph on a group G admits odd automorphisms in the left regular representation G_L of G if and only if Sylow 2-subgroups of G are cyclic. In particular,

- a Cayley graph of order $2 \pmod 4$ admits odd automorphisms.

Proposition

Let $X = Cay(G, S)$ be a Cayley graph on an abelian group G and let $\tau \in \mathrm{Aut}(G)$ be such that $\tau(i) = -i$. Then $\langle G_L, \tau \rangle \leq \mathrm{Aut}(X)$, and there exists an odd automorphism in $\langle G_L, \tau \rangle$ if and only if one of the following holds:

- $|G| \equiv 3 \pmod 4$,
- $|G| \equiv 2 \pmod 4$,
- $|G| \equiv 0 \pmod 4$ and a Sylow 2-subgroup of G is cyclic.

A graph X is called symmetric if its automorphism group acts transitively on its vertex set as well as its arc set.

arc $=$ directed edge

Cubic symmetric graphs

A symmetric graph X is said to be s-regular if for any two s-arcs in X, there is a unique automorphism of X mapping one to the other.

Tutte, 1947

Every finite cubic symmetric graph is s-regular for some $s \leq 5$.

The list of all possible pairs of vertex and edge stabilizers in cubic s-regular graphs:

s	$\mathrm{Aut}(X)_v$	$\mathrm{Aut}(X)_e$
1	$\mathbb{Z}_3$	id
2	S_3	$\mathbb{Z}_2^2$ or $\mathbb{Z}_4$
3	$S_3 \times \mathbb{Z}_2$	D_8
4	S_4	D_{16} or QD_{16}
5	$S_4 \times \mathbb{Z}_2$	$(D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$

The vertex stabilizer is of order $3 \cdot 2^{s-1}$ in a cubic s-regular graph.

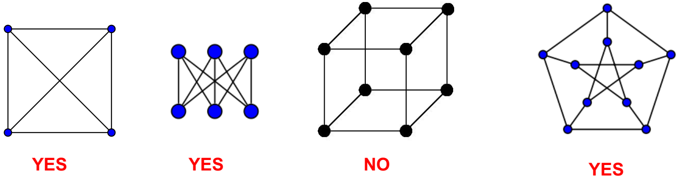
17 types of cubic symmetric graphs (Conder, Nedela, 2009)

s	Type	Bipartite?	s	Type	Bipartite?	s	Type	Bipartite?
1	$\{1\}$	Sometimes	3	$\{2^1, 3\}$	Never	5	$\{1, 4^1, 4^2, 5\}$	Always
2	$\{1, 2^1\}$	Sometimes	3	$\{2^2, 3\}$	Never	5	$\{4^1, 4^2, 5\}$	Always
2	$\{2^1\}$	Sometimes	3	$\{3\}$	Sometimes	5	$\{4^1, 5\}$	Never
2	$\{2^2\}$	Sometimes	4	$\{1, 4^1\}$	Always	5	$\{4^2, 5\}$	Never
3	$\{1, 2^1, 2^2, 3\}$	Always	4	$\{4^1\}$	Sometimes	5	$\{5\}$	Sometimes
3	$\{2^1, 2^2, 3\}$	Always	4	$\{4^2\}$	Sometimes			

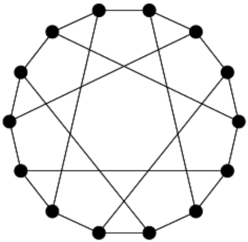
Examples: $K_4 = \{1, 2^1\}$, $K_{3,3} = \{1, 2^1, 2^2, 3\}$, $Q_3 = \{1, 2^1\}$, $F010A = \{2^1, 3\}$, $F014A = \{1, 4^1\}$, $F016A = \{1, 2^1\}$, $F018A = \{1, 2^1, 2^2, 3\}$, $F020A = \{1, 2^1\}$, $F020B = \{2^1, 2^2, 3\}$.

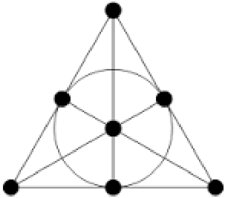
The four smallest cubic symmetric graphs

Odd automorphisms?



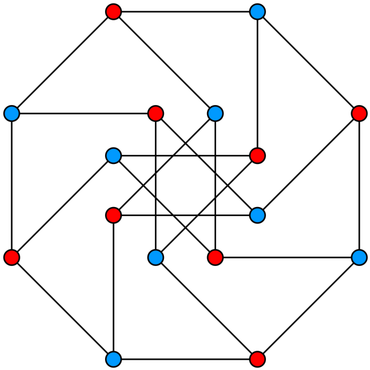
The Heawood graph $F014A$ of type $\{1, 4^1\}$

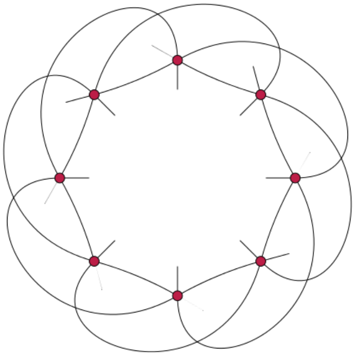




Odd automorphism? YES

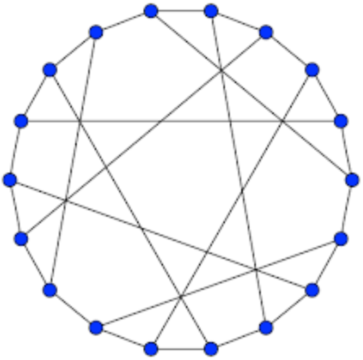
The Möbius-Kantor graph $F016A$ of type $\{1, 2^1\}$

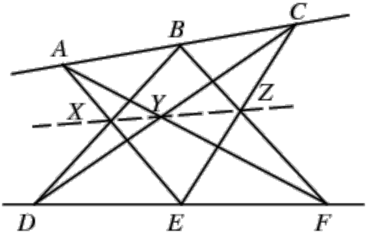




Odd automorphism? NO

The Pappus graph $F018A$ of type $\{1, 2^1, 2^2, 3\}$





Odd automorphism? YES

Automorphism group of a cubic symmetric graph

The automorphism group of any finite cubic symmetric graph is an epimorphic image of one of the following seven groups:

$$
\begin{aligned}
G_1 &= \langle h, a \mid h^3 = a^2 = 1\rangle,\\
G_2^1 &= \langle h, a, p \mid h^3 = a^2 = p^2 = 1, apa = p, php = h^{-1}\rangle,\\
G_2^2 &= \langle h, a, p \mid h^3 = p^2 = 1, a^2 = p, php = h^{-1}\rangle,\\
G_3 &= \langle h, a, p, q \mid h^3 = a^2 = p^2 = q^2 = 1, apa = q, qp = pq, ph = hp, php = h^{-1}\rangle,\\
G_4^1 &= \langle h, a, p, q, r \mid h^3 = a^2 = p^2 = q^2 = r^2 = 1, apa = p, aqa = r, h^{-1}ph = q,\\
&\quad h^{-1}qh = pq, rhr = h^{-1}, pq = qp, pr = rp, rq = pqr\rangle,\\
G_4^2 &= \langle h, a, p, q, r \mid h^3 = p^2 = q^2 = r^2 = 1, a^2 = p, a^{-1}qa = r, h^{-1}ph = q,\\
&\quad h^{-1}qh = pq, rhr = h^{-1}, pq = qp, pr = rp, rq = pqr\rangle,\\
G_5 &= \langle h, a, p, q, r, s \mid h^3 = a^2 = p^2 = q^2 = r^2 = s^2 = 1, apa = q, ara = s, h^{-1}ph = p,\\
&\quad h^{-1}qh = r, h^{-1}rh = pqr, shs = h^{-1}, pq = qp, pr = rp, ps = sp, qr = rq,\\
&\quad qs = sq, sr = pqrs\rangle.
\end{aligned}
$$

Odd automorphisms in cubic symmetric graph

Theorem: *Let X be a cubic symmetric graph of order $2n$. Then Table below gives a full information on existence of odd automorphisms in X.*

Type	Odd automorphisms exist if and only if
$\{1\}$	n odd
$\{1, 2^1\}$	n odd, or $n = 2^{k-1}(2t+1)$ and X is a (2t+1)-Cayley graph on a cyclic group of order 2^k, where $k \geq 2$
$\{2^1\}$	n odd and X bipartite
$\{2^2\}$	never
$\{1, 2^1, 2^2, 3\}$	n odd
$\{2^1, 2^2, 3\}$	n odd
$\{2^1, 3\}$	n odd
$\{2^2, 3\}$	n odd
$\{3\}$	n odd and X bipartite
$\{1, 4^1\}$	n odd
$\{4^1\}$	n odd and X bipartite
$\{4^2\}$	n odd
$\{1, 4^1, 4^2, 5\}$	n odd
$\{4^1, 4^2, 5\}$	n odd
$\{4^1, 5\}$	never
$\{4^2, 5\}$	never
$\{5\}$	n odd and X bipartite

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WELCOME TO PORTOROŽ IN JULY 2020 FOR THE 8TH EUROPEAN CONGRESS OF MATHEMATICS!

THANK YOU!

HVALA!

Спасибо!